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CONTINUITY OF WEAK SOLUTIONS TO A GENERAL POROUS MEDIA EQUATION.

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Emmanuele DiBenedetto

14 MRC-TSR-2189

15 DAAG29-80-C-0041

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

11 12 49

March 1981

(Received January 23, 1981)

Approved for public release
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U. S. Army Research Office
P. O. Box 12211
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UNIVERSITY OF WISCONSIN - MADISON
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CONTINUITY OF WEAK SOLUTIONS TO A GENERAL POROUS MEDIA EQUATION

Emmanuele DiBenedetto[(1)]

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ABSTRACT

We establish the continuity of weak solutions to singular equations of the type

$$\frac{\partial}{\partial t}\beta(u) - \mathrm{div}\,\vec{a}(x,t,u,\nabla_x u) + b(x,t,u,\nabla_x u) = 0 \quad ,$$

where $\beta(\cdot)$ is a graph satisfying assumptions appropriate for the equation of porous media, in particular for the filtration of gases.

AMS(MOS) Subject Classifications: 35K10, 35K15, 35K20, 35K65.

Key Words: singular or degenerate evolution equation, free boundary, porous media

Work Unit Number 1 - Applied Analysis

(1) Mathematics Department of Indiana University, Bloomington, Indiana 47405.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

-- in particular, for the filtration of gases.

SIGNIFICANCE AND EXPLANATION

The singular parabolic equations treated, ~~here~~ in this report serve as a model for filtration of fluids in porous media. The function $\beta(s) = |s|^{\frac{1}{m}} \text{sign } s,\ m > 1$ serves as the model situation for such problems and makes the equation singular.

Usually solutions of boundary value problems associated with such equations are found in a global sense, i.e. they are characterized as equivalence classes in certain Sobolev spaces. It is of interest to decide whether they may be defined pointwise and whether they possess some local regularity such as continuity.

In this paper we prove that global (weak) solutions are in fact continuous. Moreover, we study under what circumstances their continuity can be extended up to the boundary of the domain where the process takes place.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

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CONTINUITY OF WEAK SOLUTIONS TO A GENERAL
POROUS MEDIA EQUATION

Emmanuele DiBenedetto[†]

1. Introduction

The aim of this paper is to extend the results I obtained in a previous work [10] about the continuity of weak solutions of singular parabolic equations in divergence form of the type

$$\frac{\partial}{\partial t}\beta(u) - \operatorname{div} \vec{a}(x,t,u,\nabla_x u) + b(x,t,u,\nabla_x u) \ni 0 \tag{1.1}$$

in the sense of distributions over a domain Q in $\mathbb{R}^{N+1}$, $N \geq 1$. In [10] I considered the case of β being a coercive, maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with a jump at the origin, namely

$$\beta(s) = \begin{cases} \beta_1(s) & , \quad s > 0 \\ [-\nu, 0] & , \quad s = 0 \\ \beta_2(s) - \nu & , \quad s < 0 \end{cases}, \tag{1.2}$$

where β_i, $i = 1, 2$ are increasing coercive Lipschitzian functions in $\mathbb{R}$, and ν is a given positive constant. The situation was typical of diffusion processes with a change of phase. Here we consider the case of β continuous, coercive, monotone in $\mathbb{R}$, such that $\beta'(s)$ "blows-up" at $s = 0$. The model example of β I have in mind is

[†]Mathematics Department of Indiana University, Bloomington, Indiana 47405.

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$$(1.3) \qquad \beta(s) = |s|^{\frac{1}{m}} \operatorname{sign} s \ , \quad m > 1 \ ,$$

which occurs in filtration of gases in porous media when the flow obeys a polytropic law.

Our goal is to prove that weak solutions (in a sense to be made precise) of (1.1), for β such as (1.3) are continuous.

To our knowledge available, related regularity results essentially deal with non-negative weak solutions of

$$(1.4) \qquad \beta(u)_t - \Delta u = 0 \quad \text{in} \quad \mathcal{D}'(\Omega_T) \ ,$$

where Ω_T is a cylindrical domain in $\mathbb{R}^{N+1}$.

For $N = 1$, $\Omega_T \equiv \mathbb{R} \times \mathbb{R}^+$, the sharpest results are due to D. Aronson [1, 2, 3]. When $\beta(\cdot)$ is as in (1.3) he proves that weak solutions of (1.4) are locally Hölder continuous with respect to the space variable, with optimal exponent $m/m-1$. The time-regularity is investigated in [13, 11, 9].

For $N > 1$ and $\beta(\cdot)$ as in (1.3), continuity and Hölder continuity are due to Caffarelli and Friedman [5, 6]. Their proof employs an interesting regularizing effect of operators like in (1.4), on non-negative solutions, discovered by Aronson and Bénilan [4].

For equations bearing lower order terms we mention an unpublished result of A. Friedman, reported by L.A. Peletier in [15]: Non-negative weak-solutions of

$$(1.5) \qquad \frac{\partial}{\partial t} u - \Delta |u|^{m-1} u = -|u|^{n-1} u \ , \quad m > 1 \ , \quad n > 0$$

are continuous in their domain of definition, provided that $n \geq m$.

When the signum restriction on the solution is relaxed then the continuity of weak solutions of

$$\beta(u)_t - \Delta u = f(x,t) \quad \text{in} \quad \mathcal{D}'(\Omega_T)$$

has been proved by Caffarelli and Evans [7] in the case $f(x,t) \equiv 0$, and P. Sacks [16], if $f \not\equiv 0$, under suitable assumptions on f.

In the present paper no assumptions have been made concerning the signum of the solution, or the linearity of the operator involving space-derivative. Also no relationship has been imposed between the (possibly non-linear) $\vec{a}(x,t,u,\nabla_x u)$, $b(x,t,u,\nabla_x u)$ and the graph β .

The method of proof closely reflects the one presented in [10]. One feature that made possible the results of [10] was the observation that graphs β such as (1.2) are, roughly speaking the sum of the identity graph and the maximal monotone extension of the Heaviside function. This is not the case in the present situation, so that even remaining in the same framework, a different analysis has to be produced to take in account the nature of the singularity of β . This will result in a modulus of continuity for u which is "worse" than the one derived in [10].

Our work is organized as follows. Section 2 collects assumptions and statement of results. Some preliminary material form section 3 whereas the proof of theorem 1 is the object of sections 4, 5. Finally in section 6 we make some remarks about continuity of the solution up to the boundary.

2. Assumptions and statement of results:

We start by introducing some notation and making precise the meaning of solution of (1.1).

Let Ω be a bounded domain in $\mathbb{R}^N$ of boundary $\partial\Omega$ and for $0 < T < \infty$ let $\Omega_T \equiv \Omega \times (0,T]$, $\Omega(t) = \Omega \times \{t\}$, $S_T = \bigcup_{0<t\le T} \partial\Omega \times \{t\}$, $\Gamma = S_T \cup \Omega(0)$.

For q , $r \ge 1$, we denote by $L_{q,r}(\Omega_T)$ the Banach space of those measurable functions mapping $\Omega_T \to \mathbb{R}$, with norm defined by

$$\|u\|^r_{q,r,\Omega_T} = \int_0^T \|u\|_{q,\Omega}\,(t)\,dt\ ,$$

where $\|u\|^q_{q,\Omega}\,(t) = \int_\Omega |u(x,t)|^q\,dx$. When $q = r = 2$, $L_{2,2}(\Omega_T)$ coincides with the Hilbert space $L_2(\Omega_T)$ whose inner product $(\cdot,\cdot)_{2,\Omega_T}$ generates the norm $\|\cdot\|_{2,\Omega_T} \equiv \|\cdot\|_{2,2,\Omega_T}$. Let $W_2^{1,0}\,(\Omega_T)$ denote the Hilbert space with inner product

$$(u,v)_{W_2^{1,0}(\Omega_T)} = (u,v)_{2,\Omega_T} + \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}\,,\,\frac{\partial v}{\partial x_i}\right)_{2,\Omega_T}$$

while $W_2^{1,1}(\Omega_T)$ denotes the Hilbert space with inner product

$$(u,v)_{W_2^{1,1}(\Omega_T)} = (u,v)_{W_2^{1,0}(\Omega_T)} + \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right)_{2,\Omega_T}$$

Here $\frac{\partial u}{\partial x_i}$, $\frac{\partial u}{\partial t}$ denote generalized derivatives. With $\overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ we denote the space of those elements in $W_2^{1,1}(\Omega_T)$ whose trace on $\partial\Omega \times (0,T]$ is zero.

Let $V_2(\Omega_T) \subset W_2^{1,0}(\Omega_T)$ denote the Banach space with norm

$$|u|_{V_2(\Omega_T)} = \operatorname*{ess\,sup}_{0 \le t \le T} \|u(\cdot,t)\|_{2,\Omega}^2 + \|\nabla_x u\|_{2,\Omega_T}^2$$

where

$$\|\nabla_x u\|_{2,\Omega_T}^2 = \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_i}\right)_{2,\Omega_T}$$

Finally we let $V_2^{1,0}(\Omega_T) \subset V_2(\Omega_T)$ denote the Banach space of functions such that the map $t \to u(\cdot,t)$ is continuous with respect to $\|\cdot\|_{2,\Omega}$, and the norm $|\cdot|_{V_2^{1,0}(\Omega_T)}$ is that of $V_2(\Omega_T)$ with the ess deleted.

<u>Definition</u>: By a weak solution of (1.1) we mean a function $u \in V_2(\Omega_T)$ such that

$$(2.1) \qquad \int_\Omega \beta(u)\, \phi(x,\tau)\, dx \Big|_{t_0}^{t} + \int_{t_0}^{t} \int_\Omega \{-\beta(u)\, \phi_t + \vec{a}(x,\tau,u,\nabla_x u) \cdot \nabla_x \phi$$
$$+ b(x,\tau,u,\nabla_x u)\phi\}\, dx\, d\tau = 0 ,$$

for all $\phi \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ and almost all t_0, t such that $0 < t_0 < t \le T$.

The integrals in (2.1) are well defined modulo basic assumptions listed below. If $u \in V_2(\Omega_T)$ is solution of a boundary value problem associated with (1.1), then it satisfies (2.1), the boundary conditions being specified separately. By the local nature of our arguments, we need not associate (1.1) with a particular boundary value problem.

Throughout the paper we will make the following assumptions, on β, $\vec{a} \equiv (a_1, a_2, \ldots, a_N)$ and b.

$[A_1]$ Let $\beta(\cdot)$ be continuous, monotone increasing in $\mathbb{R}$ such that $\beta(0) = 0$. With $\beta'(s)$ we denote the Dini numbers (whenever they exist),

$$\beta'(s) = \begin{cases} \limsup\limits_{h \searrow 0} \dfrac{\beta(s) - \beta(s-h)}{h} \ , & s > 0 \\ \limsup\limits_{h \searrow 0} - \dfrac{\beta(s) - \beta(s+h)}{h} \ , & s < 0 \end{cases}$$

and on $s \to \beta'(s)$ assume the following

(i) $0 < \alpha_0 \le \beta'(s)$, $\forall s \in \mathbb{R}\backslash\{0\}$, where α_0 is a given constant.

(ii) $\liminf\limits_{|s| \to 0} \beta'(s) = +\infty$

(iii) There exists an interval $[-\delta_0 , \delta_0]$ around the origin such that $\beta'(s) \le \beta'(r)$ for $s \in \mathbb{R}\backslash[-\delta_0 , \delta_0]$ and $r \in [-\delta_0 , \delta_0]\backslash\{0\}$ and $\beta'(\cdot)$ is decreasing over $(0 , \delta_0]$ and increasing over $[-\delta_0 , 0)$.

<u>Remarks</u>: (i) We will use without mention the following consequence of assumption $[A_1]$:

$$\sup_{|s|>\delta_0} \beta'(s) \le \alpha_1 \equiv \max\{\beta'(\delta_0) \ ; \ \beta'(-\delta_0)\} \ .$$

(ii) Without loss of generality we might assume that

$$\beta'(s) > 1 \ , \ \forall s \in [-\delta_0 , \delta_0]\backslash\{0\} \ .$$

(iii) Notice that there is no symmetry requirement on $\beta(\cdot)$ around the origin.

$[A_2]$ a_i , b are measurable on $\Omega_T \times \mathbb{R}^{N+1}$ $i = 1 , 2,\ldots,N$.

$[A_3]$ $\vec{a}(x,t,u,\vec{p}) \cdot \vec{p} \ge C_0(|u|) \ |\vec{p}|^2 - \phi_0(x,t)$, $\forall \vec{p} \in \mathbb{R}^N$

$|a_i(x,t,u,\vec{p})| \le \mu_0(|u|) \ |\vec{p}| + \phi_1(x,t)$

$|b(x,t,u,\vec{p})| \le \mu_1(|u|) \ |\vec{p}|^2 + \phi_2(x,t)$

where $C_0(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and decreasing,

$\mu_i(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous and increasing, $i = 0 , 1$

and the $\phi_i \cdot i = 0 , 1 , 2$ are non-negative and satisfy

$$\phi_0 \, , \, \phi_1^2 \, , \, \phi_2 \in L_{\hat{q},\hat{r}}(\Omega_T)$$

where the numbers $\hat{q}$, $\hat{r} \geq 1$ are linked by the relations

$$\frac{1}{\hat{r}} + \frac{N}{2\hat{q}} = 1 - \kappa_1 \quad , \quad 0 < \kappa_1 < 1$$

$$\hat{q} \in \left[\frac{N}{2(1-\kappa_1)} \, , \, \infty\right] \quad , \quad \hat{r} \in \left[\frac{1}{1-\kappa_1} \, , \, \infty\right] \quad , \quad \text{for} \quad N \geq 2$$

$$\hat{q} \in (1,\infty) \quad , \quad \hat{r} \in \left(\frac{1}{1-\kappa_1} \, , \, \frac{2}{1-2\kappa_1}\right) \quad , \quad 0 < \kappa_1 < \frac{1}{2} \quad \text{for} \quad N = 1 \, .$$

With $\{\beta_n\}$ we denote a sequence of $C^\infty(\mathbb{R})$ functions such that $\beta_n \to \beta$ uniformly on compacts of $\mathbb{R}\backslash\{0\}$, and satisfying, $\beta_n(0) = 0$ and

(2.2) $$0 < \alpha_0 \leq \beta_n'(s) \leq \beta'(s) \, , \, \forall \, s \in \mathbb{R}\backslash\{0\} \, , \, n = 1 \, , 2, \ldots$$

Such a sequence can obviously be constructed.

$[A_4]$ Let $u \in V_2(\Omega_T)$ be an essentially bounded weak solution of (1.1). We assume that u can be constructed as the weak $- V_2(\Omega_T)$ limit of a sequence $\{u_n\}$ such that

$$\|u_n\|_{\infty,\Omega_T} \leq M < \infty \quad , \quad n = 1 \, , 2 \, , 3, \ldots$$

for some constant M , and each $u_n \in V_2^{1,0}(\Omega_T)$ is a weak-solution of (1.1), in the sense of identity (2.1) with $\beta(\cdot)$ replaced by $\beta_n(\cdot)$. Also since $u_n \in V_2^{1,0}(\Omega_T)$, each u_n will satisfy (2.1) for <u>all</u> intervals $[t_0,t] \subset (0,T]$.

<u>Remark</u>: Assumption $[A_4]$ is introduced to justify some of the calculations in what follows, and is not restrictive in view of the available existence theory. (See references in [10]). *We can now state our main result.*

<u>Theorem 1</u>: Let $[A_1] - [A_3]$ hold. Then any essentially bounded weak solution of (1.1), satisfying $[A_4]$, is continuous in Ω_T .

Remarks: (i) By the local nature of our arguments, in Theorem 1, the function u need not be defined in a cylindrical domain, since we can always reduce to this case by selecting in (2.1) test functions supported in cylindrical domains contained in Ω_T. Hence for the purpose of proving Theorem 1 we need only to assume that u is locally essentially bounded in Q that $u \in V_{2,loc}(Q)$, and satisfies $[A_4]$ locally.

(ii) Assumptions $[A_2]$ - $[A_3]$ are the same to those imposed in [14] to study the Hölder continuity of weak solutions of (1.1) with $\beta(s) = s$.

(iii) If in $[A_3]$ there exist $\sigma \in [\frac{N}{N+2}, 2]$ such that

$$|b(x,t,u,\vec{p})| \le \mu_1(|u|)\ |\vec{p}|^{2-\sigma} + \phi_2(x,t)\ ,$$

then the uniform local essential boundedness of the approximations u_n, follows from the results of [14].

(iv) In $[A_3]$ for the space dimension $N = 1$, one could also allow $\hat{q} = 1$ and $\hat{q} = \infty$, modulo some modifications in the proof of theorem 1. We omit the lenghty modifications needed, since for $N = 1$ more precise regularity results are available (even though for particular cases of (1.1)) [1 , 2 , 11 , 13].

If (1.1) is associated with an initial boundary value problem of Dirichlet or Neumann type, then under suitable assumptions on the boundary conditions and on the smoothness of $\partial\Omega$, the continuity of u can be extended to the closure of Ω_T. We refer to section 6 for the precise statement of these results.

We will prove the theorem in terms of the sequence $\{u_n\}$ introduced in $[A_4]$. Namely we will prove the following proposition.

<u>Proposition 1</u>: The sequence $\{u_n\}$ in $[A_4]$ is equicontinuous in Ω_T.

Also the proof of continuity of u in $\bar{\Omega}_T$ will be carried in terms of equicontinuity of the sequence $\{u_n\}$ in $\bar{\Omega}_T$.

We remark that by virtue of the smoothness of $\beta_n(\cdot)$, in view of the results of [14] each $u_n(x,t)$ is Hölder continuous in Ω_T with Hölder constant and exponent dependent on n.

As in [10] the method of proof consists in modifying suitably the parabolic version of DeGiorgi's estimates [8], as appearing in [14]. Roughly speaking we will construct for every point $(x_0, t_0) \in \Omega_T$ a family of nested shrinking cylinders where the oscillation of $(x,t) \to u_n(x,t)$ decreases according to the rules imposed by the operator in (1.1), but in a way which is independent of $n \in \mathbb{N}$.

The statement that a certain quantity or function depends upon the data, will mean that it can be determined in terms of $N, C_0(\cdot), \mu_0(\cdot), \mu_1(\cdot), \phi_i$ $i = 0, 1, 2$, $\hat{q}, \hat{r}, \kappa_1, \alpha_0, \delta_0$ and the essential bound of $|u|$ over Ω_T.

3. Preliminary material:

This section is devoted to the derivation of a system of integral inequalities which will be the main tool in the proof of Theorem 1.

Let $u \in L_{q,r}(\Omega_T)$ and $k \in \mathbb{R}$. Set $(u-k)^+ = \max\{(u-k)\,;0\}$; $(u-k)^- = \max\{-(u-k)\,;0\}$. It is obvious that $(u-k)^{\pm} \in L_{q,r}(\Omega_T)$ and it is known that if $u \in V_2^{1,0}(\Omega_T)$, then also $(u-k)^{\pm}$ belong to $V_2^{1,0}(\Omega_T)$, ([14]).

With $B(R)$ we denote a ball of radius R in $\mathbb{R}^N$. and if $x \to u(x)$ is defined in $B(R)$ we set

$$A^+_{k,R} \equiv \{x \in B(R) \,|\, u(x) > k\} \qquad A^-_{k,R} \equiv \{x \in B(R) \,|\, u(x) < k\}.$$

Also let κ_N denote the measure of the surface of the unit ball in $\mathbb{R}^N$, so that meas $B(R) = \kappa_N R^N$. The Steklov averagings u_h of $u \in L_{q,r}(\Omega_T)$ are defined as

$$u_h(x,t) = \frac{1}{h}\int_t^{t+h} u(x,\tau)\,d\tau\,, \quad 0 \le t \le T - h\,, \quad h > 0\,.$$

If $u \in V_2^{1,0}(\Omega_T)$, then $u_h \to u$ in the topology of $V_2^{1,0}(\Omega_{T-\varepsilon})$, $\forall \varepsilon > 0$.

An integration by parts shows (see [14] for details) that if $u_n \in V_2^{1,0}(\Omega_T)$ satisfies identity (2.1) with β replaced by β_n, then

$$(3.1) \qquad \int_{t_0}^{t} \int_{\Omega} \left\{ \frac{\partial}{\partial t} \Big[\beta_n(u_n)\Big]_h \phi + \Big[\vec{a}(x,\tau,u_n,\nabla_x u_n\Big]_h \cdot \nabla_x \phi + \Big[b(x,\tau,u_n,\nabla_x u_n\Big]_h \phi \right\} dx\, d\tau = 0$$

for all $\phi \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ and all intervals $[t_0,t] \subset (0, T-h]$.

From now on $\{(x,t) \to u_n(x,t)\}$ will denote a sequence satisfying (3.1) and M is a positive number such that

$$\|u_n\|_{\infty,\Omega_T} \le M , \quad n = 1, 2, 3, \ldots$$

A system of integral inequalities will be derived for $(x,t) \to u_n(x,t)$ by making particular selections of the test functions ϕ in (3.1) and letting $h \to 0$. Next we construct test functions in (3.1).

Let σ_1, $\sigma_2 \in (0,1)$ and consider the concentric balls $B(R)$ and $B(R-\sigma_1 R)$, and the cylinders $Q(R,\lambda) \equiv B(R) \times [t_0, t_0+\lambda]$, and $Q(R-\sigma_1 R, \lambda-\sigma_2\lambda) \equiv B(R-\sigma_1 R) \times [t_0+\sigma_2\lambda, t_0+\lambda]$, $\lambda > 0$.

Define cutoff functions in $Q(R,\lambda)$ as follows:

(a) $\zeta \in C_\circ[Q(R,\lambda)]$, such that $\zeta(x,t)|_{\partial B(R)} = 0, \forall t \in [t_0, t_0+\lambda]$, $\zeta(x,t_0) = 0$, $\forall x \in B(R)$ and $\zeta(x,t) = 1$ $(x,t) \in Q(R-\sigma_1 R, \lambda-\sigma_2\lambda)$, $\frac{\partial}{\partial t}\zeta \ge 0$, $|\nabla_x \zeta| \le (\sigma_1 R)^{-1}$; $|\frac{\partial}{\partial t}\zeta| \le (\sigma_2\lambda)^{-1}$.

(b) $\zeta \in C_0(B(R))$ such that $\zeta(x) = 1$, $x \in B(R-\sigma_1 R)$, $|\nabla \zeta| \le (\sigma_1 R)^{-1}$.

For any cylinder $Q(R,\lambda) \subset \Omega_{T-h}$ we make the following selection of test function in (3.1),

$$\phi = \pm(\beta_n^{-1}([\beta_n(u_n)]_h) - k)^{\pm} \zeta^2 ,$$

where $k \in \mathbb{R}$ satisfies

$$(3.2) \qquad \operatorname*{ess\,sup}_{Q(R,\lambda)} (u_n - k)^{\pm} \le \delta ,$$

for some $\delta > 0$ to be selected, and $(x,t) \to \zeta(x,t)$ is chosen as in (a).

We treat the term involving $\frac{\partial}{\partial t}[\beta_n(u_n)]_h$ in (3.1) as follows. (We drop the subscript n for simplicity of notation)

$$I_h = \int_{t_0}^{t} \int_{\Omega} \pm \frac{\partial}{\partial t} [\beta(u)]_h \, [\beta^{-1}([\beta(u)]_h) - k]^{\pm} \, \zeta^2(x,\tau) \, dx \, d\tau =$$

$$= \int_{t_0}^{t} \int_{\Omega} \zeta^2(x,\tau) \frac{\partial}{\partial t} \Lambda_h(x,\tau) \, dx \, d\tau ,$$

where

$$\Lambda_h = \pm \int_0^{[\beta(u)]_h} [\beta^{-1}(\xi) - k]^{\pm} \, d\xi .$$

Hence

$$I_h = \int_{\Omega} \Lambda_h(x,t) \, \zeta^2(x,t) \, dx - \int_{t_0}^{t} \int_{\Omega} \Lambda_h(x,\tau) \frac{\partial}{\partial t} \zeta^2(x,\tau) \, dx \, d\tau ,$$

so that letting $h \to 0$ we obtain

$$I_h \to I \geq \frac{\alpha_0}{2} \int_{\Omega} (u-k)^{\pm^2} \, \zeta^2(x,t) \, dx - \int_{t_0}^{t} \int_{\Omega} \Lambda(x,\tau) \frac{\partial}{\partial t} \zeta^2(x,\tau) \, dx \, d\tau .$$

In estimating the remaining terms in (3.1) we use assumptions $[A_1] - [A_3]$, (3.2) and routine calculations to obtain (see [10] for details)

(3.3) $$\int_{t_0}^{t} \int_{\Omega} \{\vec{a}(x,\tau,u,\nabla_x u) \cdot \nabla_x \phi + b(x,\tau,u,\nabla_x u)\phi\} \, dx \, d\tau \geq$$

$$\geq (C_0(M) - \epsilon - \delta \, \mu_1(M)) \int_{t_0}^{t} |\nabla_x (u-k)^{\pm}|^2 \, \zeta^2(x,\tau) \, dx \, d\tau -$$

$$- (\epsilon^{-1} \mu_0^2(M) + 1) \int_{t_0}^{t} \int_{\Omega} [(u-k)^{\pm}]^2 \, |\nabla_x \zeta|^2 \, dx \, d\tau -$$

$$- \int_{t_0}^{t} \int_{\Omega} [\phi_0 + \delta\phi_2 + \phi_1^2] \, \zeta^2(x,\tau) \, \chi[(u-k)^{\pm} > 0] \, dx \, d\tau .$$

Here $\chi(\Sigma)$ denotes the characteristic function of the set Σ. By Hölder's inequality

$$J = \int_{t_0}^{t} \int_{\Omega} [\phi_0 + \delta\phi_2 + \phi_1^2] \, \zeta^2(x,\tau) \, \chi[(u-k)^{\pm} > 0] \, dx \, d\tau \leq$$

$$\max\{1,\delta\}\ \|\phi_0+\phi_2+\phi_1^2\|_{\hat q,\hat r,\Omega_T}\left\{\int_{t_0}^{t}\left[\mathrm{meas}\ A^{\pm}_{k,R}(\tau)\right]^{\frac{\hat q-1}{\hat q}\frac{\hat r}{\hat r-1}}d\tau\right\}^{\frac{\hat r-1}{\hat r}} .$$

Since $\frac{1}{\hat r}+\frac{N}{2\hat q}=1-\kappa_1$, the numbers

$$q=\frac{2\hat q(1+\kappa)}{\hat q-1}\ ,\quad r=\frac{2\hat r(1+\kappa)}{\hat r-1}\ ,\quad \kappa=\frac{2\kappa_1}{N}$$

satisfy the relationship

$$\frac{1}{r}+\frac{N}{2q}=\frac{N}{4}\ , \tag{3.4}$$

and their admissible range is

$$\begin{aligned} &q\in\left(2\ ,\ \frac{2N}{N-2}\right]\ , && r\in[2,\infty) && \text{for}\quad N\ge 3\\ &q\in(2,\infty)\ , && r\in(2,\infty) && \text{for}\quad N=2\\ &q\in(2,\infty)\ , && r\in[4,\infty) && \text{for}\quad N=1\ . \end{aligned} \tag{3.5}$$

With this notation we have

$$J\le\max\{1,\delta\}\ \|\phi_0+\phi_2+\phi_1^2\|_{\hat q,\hat r,\Omega_T}\left\{\int_{t_0}^{t}\left[\mathrm{meas}\ A^{\pm}_{k,R}(\tau)\right]^{\frac{r}{q}}d\tau\right\}^{\frac{2}{r}(1+\kappa)} .$$

On the right-hand side of (3.3) we select $\varepsilon=\frac{C_0(M)}{4}$ and

$$\delta=\min\left\{1\ ,\ \delta_0\ ,\ \frac{C_0(M)}{4\mu_1(M)}\right\} \tag{3.6}$$

where δ_0 is the number in $[A_1]$. Therefore collecting all the previous estimates we obtain the inequalities

$$\begin{aligned}\alpha_0\|(u-k)^{\pm}\zeta\|^2_{2,\Omega}(t)+C_0(M)\int_{t_0}^{t}\int_\Omega|\nabla_x(u-k)^{\pm}|^2\zeta^2\,dx\,d\tau\le\\ \gamma\int_{t_0}^{t_0+\lambda}\int_\Omega[(u-k)^{\pm}]^2\left(|\nabla_x\zeta|^2+\zeta|\zeta_t|\right)dx\,d\tau\end{aligned} \tag{3.7}$$

$$+ \gamma\left\{\int_{t_0}^{t_0+\lambda} [\text{meas } A^{\pm}_{k,R}(\tau)]^{\frac{r}{q}}\, d\tau\right\}^{\frac{2}{r}(1+\kappa)}$$

$$+ \gamma \int_{t_0}^{t}\int_{\Omega} \Lambda\,(x,\tau)\,\frac{\partial}{\partial t}\,\zeta^2\,(x,\tau)\,dx\,d\tau \quad , \quad \forall\, t \in [t_0\,, t_0+\lambda]\ ,$$

where γ is a constant depending only upon the data. We stress the fact that the choice of ε , δ and γ is independent of n .

The inequalities above are valid for every cylinder $Q(R,\lambda) \subset \Omega_T$ and every $k \in \mathbb{R}$ satisfying (3.2) with the choice (3.6) of the parameter δ .

A change of variable in the integral defining $\Lambda\,(x,t)$ gives

$$\Lambda\,(x,t) = \int_0^{\pm(u_n-k)^{\pm}} \eta\, \beta_n'(k+\eta)\, d\eta \tag{3.8}$$

Therefore

$$\Lambda\,(x,t) \le \beta_n(M)(u_n-k)^{\pm} \le \max\{\beta(M)\ ,\ \beta(-M)\}(u_n-k)^{\pm}$$

As a consequence, recalling the construction of $(x,t) \to \zeta(x,t)$, and suitably redefining the constant γ , we have the inequalities

$$|(u_n-k)^{\pm}|^2_{V_2^{1,0}[B(R-\sigma_1R)\times(t_0+\sigma_2\lambda,t_0+\lambda)]} \le \gamma\,[(\sigma_1R)^{-2} + (\sigma_2\lambda)^{-1}]\ \|(u_n-k)^{\pm}\|^2_{2,Q(R,\lambda)} \tag{3.9}$$

$$+ \gamma\left\{\int_{t_0}^{t_0+\lambda}\left[\text{meas } A^{n\pm}_{k,R}\ (\tau)\right]^{\frac{r}{q}} d\tau\right\}^{\frac{2}{r}(1+\kappa)} + \gamma(\sigma_2\lambda)^{-1}\iint_{Q(R,\lambda)} (u_n-k)^{\pm}\, dx\, d\tau$$

where $A^{n\pm}_{k,R}\ (t) = \{x \in B(R)\,|\,(u_n-k)^{\pm}(x,t) > 0\}$

Inequalities (3.9) are valid for every $k \in \mathbb{R}$, satisfying (3.2), every cylinder $Q(R,\lambda) \subset \Omega_T$ and every $\sigma_1\ ,\ \sigma_2 \in (0,1)$. The constant γ does not depend upon n .

Suppose now that (3.7) are written for the functions $(u_n-k)^+$ for $k > 0$, then $\Lambda(x,t)$ in (3.8) can be estimated as follows.

$$\Lambda\,(x,t) \le \frac{1}{2}\sup_{s\ge k}\beta_n'(s)(u_n-k)^{+2} \le \frac{1}{2}\sup_{s\ge k}\beta'(s)(u_n-k)^{+2}$$

Therefore by redefining the constant γ , (3.7) imply

$$(3.10)\qquad |(u_n-k)^+|^2_{V_2^{1,0}[B(R-\sigma_1 R)\times(t_0+\sigma_2\lambda,\, t_0+\lambda)]}$$

$$\le \gamma \sup_{s\ge k} \beta'(s)[(\sigma_1 R)^{-2}+(\sigma_2\lambda)^{-1}]\|(u_n-k)^+\|^2_{2,Q(R,\lambda)}$$

$$+\gamma\left\{\int_{t_0}^{t_0+\lambda}[\text{meas } A^{n+}_{k,R}(\tau)]^{\frac{r}{q}}\,d\tau\right\}^{\frac{2}{r}(1+\kappa)}\quad,\quad \forall n\in\mathbb{N}\,.$$

Inequalities (3.10) are valid for all $k>0$ satisfying (3.2), all cylinder $Q(R,\lambda)\subset\Omega_T$ and all σ_1 , $\sigma_2\in(0,1)$. The constant γ does not depend upon n .

An analogous argument can be produced for the functions $(u_n-k)^-$, $k<0$. It yields inequalities like (3.10), to which we will refer to as $(3.10)^-$. All the subsequent arguments will be carried over cylinders of the form $Q(R\,,\theta R^2)\equiv B(R)\times[t_0\,,t_0+\theta R^2]$, $\theta>0$.

<u>Lemma 3.1</u>: Let $x\to\zeta(x)$ be a cutoff function chosen as in (b). Then there exists a constant $\hat{C}(\theta)$ depending upon the data, but not upon n , such that

$$\iint_{Q(R,\theta R^2)} |\nabla_x u_n|^2\,\zeta^2(x)\,dx\,d\tau \le \frac{\hat{C}(\theta)}{\sigma_1^2}\,\kappa_N\,R^N\,.$$

<u>Proof</u>: In (3.1) select the test function $\phi(x,t)=e^{\lambda\beta_n^{-1}([\beta_n(u_n)]_h)}\,\zeta^2(x)$, $\lambda>0$ to be chosen.

Routine calculations and limiting processes as $h\to 0$ yield the result. Similar estimates in an analogous situation have been carried on in [10] to which we refer for details.

<u>Lemma 3.2</u>: Let $k\in\mathbb{R}^+$, $\mu\ge\sup_{Q(R,\theta R^2)}(\beta_n(u_n)-k)^+$ and $0<\eta<\mu$. Set

$$\psi_n(x,t)=\ell n^+\left[\frac{\mu}{\mu-(\beta_n(u_n)-k)^++\eta}\right]=\max\left\{\ell n\left[\frac{\mu}{\mu-(\beta_n(u_n)-k)^++\eta}\right];\,0\right\}.$$

Then there exists a constant $C(\theta)$ independent of n , such that for all $t\in[t_0\,,t_0+\theta R^2]$

$$\int_{B(R-\sigma_1 R)}\psi_n^2(x,t)\,dx\le\int_{B(R)}\psi_n^2(x,t_0)\,dx\;+$$

$$+ \frac{C(\theta)}{a_1^2} \sup_{s \ge \beta^{-1}(k)} f'(s) \left(1 + \ln \frac{\mu}{\eta}\right)\left(1 + \frac{R^{N\kappa}}{\eta^2}\right) \kappa_N R^N .$$

<u>**Remark**</u>: For simplicity of notation we will employ the same symbol ψ_n for $\psi_n(x,t)$ and $\tilde{\psi}_n(\beta_n(u_n))$. In what follows ψ_n' will mean $\frac{\partial}{\partial v} \tilde{\psi}_n(v)$.

<u>**Proof**</u>: In (2.1) we select $\phi = (\psi^2_{n,h})' \zeta^2(x)$ where $\zeta(x)$ is as in (b) and

$$\psi_{n,h} = \ln^+ \left[\frac{\mu}{\mu - ([\beta_n(u_n)]_h - k)^+ + \eta} \right] .$$

It is apparent that $\phi \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ and that

$$(\psi_n^2)'' = 2(1 + \psi_n)(\psi_n')^2$$

The first term gives

$$\int_{t_0}^{t} \int_\Omega \frac{\partial}{\partial t} [\beta_n(u_n)]_h \, (\psi^2_{n,h})' \, \zeta^2(x) \, dx \, d\tau = \int_\Omega \psi^2_{n,h} \, (x,\tau) \, \zeta^2(x) \, dx \, \Big|_{t_0}^{t}$$

and letting $h \to 0$, for all $t \in [t_0 , t_0 + \theta R^2]$, we have

$$\int_{t_0}^{t} \int_\Omega \frac{\partial}{\partial t} [\beta_n(u_n)]_h \, (\psi^2_{n,h})' \, \zeta^2(x) \, dx \, d\tau \to \int_\Omega \psi^2_n \, (x,\tau) \, \zeta^2(x) \, dx \, \Big|_{t_0}^{t}$$

In order to treat the other terms we first let $h \to 0$ and estimate the integral so obtained in the following manner, (we drop the subscript n for simplicity).

$$\int_{t_0}^{t} \int_\Omega \{\vec{a}(x,\tau,u,\nabla_x u) \, 2(1+\psi)(\psi')^2 \, \beta'(u) \, \nabla_x u \, \zeta^2(x)$$

$$+ 2 \, \vec{a}(x,\tau,u,\nabla_x u)(\psi^2)' \, \zeta(x) \, \nabla \, \zeta(x)\} \, dx \, d\tau$$

$$\ge \int_{t_0}^{t} \int_\Omega 2 \, C_0(M)(1+\psi)(\psi')^2 \, \beta'(u) \, |\nabla_x u|^2 \, \zeta^2(x) \, dx \, d\tau$$

$$- 2 \int_{t_0}^{t} \int_\Omega \{\phi_0(x,\tau)(1+\psi)(\psi')^2 \, \beta'(u) \, \zeta^2(x)\} \, dx \, d\tau$$

$$- \frac{4}{\sqrt{\alpha_0}} \int_{t_0}^{t} \int_\Omega \mu_0(M) \, |\nabla_x u| \, \psi\psi' \, \sqrt{\beta'(u)} \, \zeta(x) \, |\nabla\zeta| \, dx \, d\tau$$

$$- 4 \int_{t_0}^{t} \int_{\Omega} \phi_1 \psi \psi' \zeta(x) |\nabla \zeta| \, dx \, d\tau$$

$$\geq [2 c_0(M) - \epsilon] \int_{t_0}^{t} \int_{\Omega} (1+\psi)(\psi')^2 \beta'(u) |\nabla_x u|^2 \zeta^2(x) \, dx \, d\tau$$

$$- 2 \int_{t_0}^{t} \int_{\Omega} \{\phi_0(x,\tau)(1+\psi)(\psi')^2 \beta'(u) \zeta^2(x)\} \, dx \, d\tau$$

$$- \frac{[4 \mu_0(M)]^2}{\epsilon \alpha_0} \int_{t_0}^{t} \int_{\Omega} \psi |\nabla \zeta|^2 \, dx \, d\tau$$

$$- 2 \int_{t_0}^{t} \int_{\Omega} \phi_1^2 (\psi')^2 \zeta^2(x) \psi \, dx \, d\tau - 2 \int_{t_0}^{t} \int_{\Omega} \psi |\nabla \zeta|^2 \, dx \, d\tau .$$

For the lower order terms we have

$$\int_{t_0}^{t} \int_{\Omega} b(x,\tau,u,\nabla_x u)(\psi^2)' \zeta^2(x) \, dx \, d\tau$$

$$\leq \mu_1(M) \int_{t_0}^{t} \int_{\Omega} |\nabla_x u|^2 (\psi^2)' \zeta^2(x) \, dx \, d\tau$$

$$+ \int_{t_0}^{t} \int_{\Omega} \phi_2(x,\tau)(\psi^2)' \zeta^2(x) \, dx \, d\tau$$

$$\leq \epsilon \int_{t_0}^{t} \int_{\Omega} |\nabla_x u|^2 (1+\psi)(\psi')^2 \beta'(u) \zeta^2(x) \, dx \, d\tau$$

$$+ \frac{4 \mu_1^2(M)}{\epsilon \alpha_0} \int_{t_0}^{t} \int_{\Omega} \psi |\nabla_x u|^2 \zeta^2(x) \, dx \, d\tau$$

$$+ 2 \int_{t_0}^{t} \int_{\Omega} \phi_2(x,\tau) \psi \psi' \zeta^2(x) \, dx \, d\tau .$$

Collecting the estimates above gives

$$\text{(3.11)} \quad \int_{B(R)} \psi^2(x,t)\zeta^2(x) \, dx + 2[C_0(M) - \epsilon] \int_{t_0}^{t} \int_{\Omega} (1+\psi)(\psi')^2 \beta'(u) |\nabla_x u|^2 \zeta^2(x) \, dx \, d\tau \leq$$

$$\leq \int_{B(R)} \psi^2(x,t_0)\zeta^2(x) \, dx + 2 \max\{1;\alpha_0^{-1}\} \int_{t_0}^{t} \int_{\Omega} (\psi')^2 (1+\psi) \beta'(u) \{\phi_0 + \phi_1^2 + \phi_2\} \zeta^2(x) \, dx \, d\tau$$

$$+ (\alpha_0 \epsilon)^{-1} \max\{[4\mu_0(M)]^2 + 2 ; 4\mu_1(M)^2\} \int_{t_0}^{t} \int_{\Omega} \psi\{|\nabla_x u|^2 \zeta^2(x) + |\nabla \zeta|^2\} dx \, d\tau$$

Now we observe that $\psi_n'(x,t)$ vanishes on the set $[(x,t) \in Q(R,\theta R^2) \mid u_n(x,t) < \beta_n^{-1}(k)]$. Moreover $\psi' \le \frac{1}{\eta}$ and $\psi \le \ln \frac{\mu}{\eta}$, so that

$$(1+\psi)(\psi')^2 \beta'(u) \le \frac{1}{\eta^2}(1+\ln\frac{\mu}{\eta}) \sup_{s>\beta_n^{-1}(k)} \beta'(s)$$

Using the definition of $\zeta(x)$ and lemma 3.1 we have

$$\int_{t_0}^{t}\int_{\Omega} \psi\{|\nabla_x u|^2 \zeta^2(x) + |\nabla \zeta|^2\}\,dx\,d\tau \le \frac{\overline{C}(\theta)}{\sigma_1^2}\left(\ln\frac{\mu}{\eta}\right)\kappa_N R^N .$$

By assumptions $[A_3]$ and Hölder's inequality

$$\int_{t_0}^{t}\int_{\Omega} \{\phi_0 + \phi_1^2 + \phi_2\}\zeta^2(x)\,dx\,d\tau \le$$

$$\le \|\phi_0+\phi_1^2+\phi_2\|_{\hat{r},\hat{q},\Omega_T}\ \theta^{\frac{2}{r}(1+\kappa)}\ \kappa_N^{\frac{2}{q}(1+\kappa)-1}\ \kappa_N R^N R^{N\kappa} ,$$

Finally choosing $\varepsilon = C_0(M)$ in (3.11) we obtain

$$\int_{B(R-\sigma_1 R)} \psi_n^2(x,t)\,dx \le \int_{B(R)} \psi_n^2(x,t_0)\,dx +$$

$$+ \frac{C(\theta)}{\sigma_1^2} \sup_{s\ge\beta_n^{-1}(k)} \beta_n'(s)\left(1+\ln\frac{\mu}{\eta}\right)\left(1+\frac{R^{N\kappa}}{\eta^2}\right)\kappa_N R^N .$$

where $C(\theta)$ depends only upon the data but not upon η .

<u>Remarks</u>: (i) If $k < 0$ and $\bar{\mu} \ge \operatorname{ess\,sup}_{Q_R^\theta} (\beta_n(u_n) - k)^-$, then an analogous lemma holds for

$$\bar{\psi}_n(x,t) = \ln^+\left[\frac{\bar{\mu}}{\bar{\mu} - (\beta_n(u_n)-k)^- + \eta}\right] ; \quad 0 < \eta < \bar{\mu} .$$

(ii) The proof shows that $C(\theta)$ increases with θ . In what follows we will use Lemma 3.2 with $0 < \theta \le 1$ and $C(\theta)$ replaced by $C(1)$.

We conclude this section by stating a lemma which we will use as we proceed, and whose proof can be found in [8 , 14].

Lemma 3.3 (De Giorgi): Let $u \in W_1^1(B(R))$ and let k, ℓ be real numbers such that $\ell > k$. Then

$$(3.12)\quad (\ell - k)\,\mathrm{meas}\, A_{\ell,R} \le D \frac{R^{N+1}}{\mathrm{meas}(B(R)\setminus A_{k,R})} \int_{A_{k,R}\setminus A_{\ell,R}} |\nabla u|\, dx ,$$

where D is a constant depending only upon the dimension N.

4. The main Proposition:

All the arguments in this section will be carried for n arbitrary but fixed. Let $(x_0, t_0) \in \Omega_T$, $t_0 > 0$ and for $R > 0$, Q_R will denote the cylinder

$$Q_R \equiv \{|x - x_0| < R\} \times [t_0 - R^2, t_0] .$$

Let $R_0 < \frac{1}{2}$ be so small that $Q_{2R_0} \subset \Omega_T$, set

$$\mu^+ = \sup_{Q_{2R_0}} u_n \;;\; \mu^- = \inf_{Q_{2R_0}} u_n \quad , \quad n \text{ fixed} ,$$

and denote with ω any positive number such that

$$2M \ge \omega \ge \operatorname*{osc}_{Q_{2R_0}} u_n = \mu^+ - \mu^- .$$

For $k \in \mathbb{R}$ and $0 < R \le 2R_0$ we set

$$Q_R^+ (k) \equiv \{(x,t) \in Q_R \mid u_n(x,t) > k\}$$

$$Q_R^- (k) \equiv \{(x,t) \in Q_R \mid u_n(x,t) < k\} .$$

In order to simplify the symbolism, since $n \in \mathbb{N}$ is fixed we will write $A^{\pm}_{k,R}(t)$ instead of $A^{n\pm}_{k,R}(t)$.

Finally we let s denote the smallest positive integer such that

$$(4.1)\qquad \frac{2M}{2^s} < \delta \le \delta_0 \quad , \quad s \ge 2 ,$$

where δ, is the number introduced in (3.6) and δ_0 is the number in assumption $[A_1]$.

Proposition 4.1: Let ω be any positive number such that

$$2M \geq \omega \geq \operatorname*{osc}_{Q_{2R_0}} u_n .$$

There exist numbers $\xi_* < 1$, $h > 1$ and a function $\pi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\operatorname*{osc}_{Q_{R_*}} u_n \leq \omega\,(1 - 2^{-\pi(\omega)}) ,$$

provided that

$$\omega\, 2^{-\pi(\omega)} \geq (2R_0)^{\frac{N\kappa}{2}} ,$$

where $R_* = \xi_*\,(2R_0)^h$, and π is given by

$$\pi(y) = s + 5 + A\,\frac{\left[\beta'\left(\frac{y}{2^s}\right) + \beta'\left(-\frac{y}{2^s}\right)\right]^c}{y^a} + \log_2 \frac{\left[\beta'\left(\frac{y}{2^s}\right) + \beta'\left(-\frac{y}{2^s}\right)\right]}{\alpha_0} .$$

Here A, a, c ξ_*, h are positive constants which depend uniquely upon the data and do not depend upon n, ω nor R_0.

Without loss of generality we may assume that

$$|\mu^-| \leq \mu^+ . \tag{4.2}$$

If the reverse inequality holds the arguments are similar. Also we will assume that

$$\operatorname*{osc}_{Q_{2R_0}} u_n = \mu^+ - \mu^- > \frac{\omega}{2^{s-1}} , \tag{4.3}$$

and treat later the case $\mu^+ - \mu^- \leq \frac{\omega}{2^{s-1}}$.

Notice that (4.2) - (4.3) imply that

$$\mu^+ - \frac{\omega}{2^s} > \left|\frac{\omega}{2^s} + \mu^-\right| \geq 0 \quad , \quad \mu^+ - \frac{\omega}{2^s} > \frac{\omega}{2^s} . \tag{4.4}$$

Observe moreover that we may assume

$$H \equiv \sup_{Q_{R_0}} \left(u_n - \left(\mu^- + \frac{\omega}{2^s}\right)\right)^- > \frac{\omega}{2^{s+1}} . \tag{4.5}$$

Indeed if (4.5) is violated

$$-\inf_{Q_{R_0}} u_n \le -\mu^- - \frac{\omega}{2^s} + \frac{\omega}{2^{s+1}}$$

and adding $\sup_{Q_{R_0}} u_n$ on the left-hand side and μ^+ on the right-hand side we obtain

$$\operatorname{osc}_{Q_{R_0}} u_n \le \omega(1 - \frac{1}{2^{s+1}})$$

and Proposition 4.1 becomes trivial.

Proposition 4.1 will be a consequence of a series of lemmas which we state and prove independently.

<u>Lemma 4.1</u>: There exists a number c_0 depending only upon the data and independent of n, ω and R_0, such that if

$$\text{meas } Q^-_{R_0}(\mu^- + \frac{\omega}{2^s}) \le c_0\, \omega^{\frac{2\kappa_1}{N+2\kappa_1}} \kappa_N R_0^{N+2},$$

then either

(i) $H = \operatorname{ess\,sup}_{Q_{R_0}} (u_n - (\mu^- + \frac{\omega}{2^s}))^- \le R_0^{\frac{N\kappa}{2}}$

or

(ii) $\text{meas } Q^-_{\frac{R_0}{2}}(\mu^- + \frac{\omega}{2^s} - \frac{1}{2}H) = 0$.

The proof of this lemma is based on inequalities (3.9) and is almost identical to the proof of lemma 3.1 of [10].

Here we only remark that without loss of generality we might assume

$$c_0\, \omega^b \le \frac{1}{2}$$

where

$$b = \frac{2\kappa_1}{N+2\kappa_1}$$

Suppose now that the assumptions of lemma 4.1 fail. Then since

$$\mu^+ - \frac{\omega}{2^s} \ge \left|\frac{\omega}{2^s} + \mu^-\right| > 0$$

we have that

$$\text{meas } Q^+_{R_0}\left(\mu^+ - \frac{\omega}{2^s}\right) \le (1 - c_0\,\omega^b)\,\kappa_N\,R_0^{N+2}$$

To shorten the notation we set

$$\theta_0 = c_0\,\omega^b\ .$$

<u>Lemma 4.2</u>: For every $\alpha \in (0,\theta_0)$ there exists $\tau \in [t_0 - R_0^2\,,\,t_0 - \alpha R_0^2]$, such that

$$\text{meas } A^+_{\mu^+ - \frac{\omega}{2^s}, R_0}(\tau) \le \frac{1-\theta_0}{1-\alpha}\,\kappa_N\,R_0^N\ .$$

<u>Proof</u>: If not, for all $\tau \in [t_0 - R_0^2\,,\,t_0 - \alpha R_0^2]$

$$\text{meas } A^+_{\mu^+ - \frac{\omega}{2^s}, R_0}(\tau) > \frac{1-\theta_0}{1-\alpha}\,\kappa_N\,R_0^N$$

and

$$\text{meas } Q^+_{R_0}\left(\mu^+ - \frac{\omega}{2^s}\right) \ge \int_{t_0 - R_0^2}^{t_0 - \alpha R_0^2} \frac{1-\theta_0}{1-\alpha}\,\kappa_N\,R_0^N\,dt > (1-\theta_0)\,\kappa_N\,R_0^{N+2}\ .$$

We will choose

$$\alpha = \frac{\theta_0}{2}$$

and observe that the previous lemma holds if $\mu^+ - \frac{\omega}{2^s}$ is replaced by $\mu^+ - \frac{\omega}{2^p}$, $\forall\ p \ge s$.

<u>Lemma 4.3</u>: Consider the cylinder

$$Q^\alpha_{R_0} \equiv \{|x - x_0| < R_0\} \times [t_0 - \alpha R_0^2\,,\,t_0]\ .$$

There exists $p_0 \in \mathbb{N}$ dependent upon α (and hence ω) such that if

$$\frac{\omega}{2^{p_0}} \ge R_0^{\frac{N\kappa}{2}}\ ,$$

then

$$\operatorname{meas}\left\{x \in B(R) \;\middle|\; \beta_n(u_n(x,t)) > \beta_n(\mu^+) - \alpha_0 \frac{\omega}{2^{p_0}}\right\} \le \left[1 - \left(\frac{\theta_0}{2}\right)^2\right] \kappa_N R_0^N$$

for all $t \in [t_0 - \alpha R_0^2, t_0]$.

<u>Proof of Lemma 4.3</u>: By monotonicity and coercivity of $\beta_n(\cdot)$

$$[u_n > \mu^+ - \frac{\omega}{2^s}] \equiv [\beta_n(u_n) > \beta_n(\mu^+ - \frac{\omega}{2^s})] \ .$$

Now $\qquad \beta_n(\mu^+ - \frac{\omega}{2^s}) \le \beta_n(\mu^+) - \alpha_0 \frac{\omega}{2^s}$, therefore

$$[\beta_n(u_n) > \beta_n(\mu^+) - \alpha_0 \frac{\omega}{2^s}] \subset [u_n > \mu^+ - \frac{\omega}{2^s}]$$

so that by lemma 4.2, there exists $\tau \in [t_0 - R_0^2, t_0 - \alpha R_0^2]$, such that

$$\operatorname{meas}\left\{[x \in B(R) : \beta_n(u_n) > \beta_n(\mu^+) - \alpha_0 \frac{\omega}{2^s}] \times (\tau)\right\} \le \frac{1-\theta_0}{1-\alpha} \kappa_N R_0^N \ .$$

We use lemma 3.2 applied to the function $(x,t) \to [\beta_n(u_n) - (\beta_n(\mu^+) - \alpha_0 \frac{\omega}{2^s})]^+ (x,t)$, in the cylinder

$$Q^\tau_{R_0} \equiv \{|x - x_0| < R_0\} \times [\tau, t_0] \ ,$$

for $\mu = \alpha_0 \frac{\omega}{2^s} \ge \sup\limits_{Q^\tau_{R_0}} [\beta_n(u_n) - (\beta_n(\mu^+) - \alpha_0 \frac{\omega}{2^s})]^+$, and $\eta = \alpha_0 \frac{\omega}{2^p}$, $p > s + 2$.

Here $t_0 - R_0^2 \le \tau \le t_0 - \alpha R_0^2$ is the number claimed by lemma 4.2.

We have

$$(4.6) \qquad \int_{B(R_0 - \sigma_1 R_0)} \ln^{+2}\left[\frac{\alpha_0 \frac{\omega}{2^s}}{\alpha_0 \frac{\omega}{2^s} - [\beta_n(u_n) - (\beta_n(\mu^+) - \alpha_0 \frac{\omega}{2^s})]^+ + \alpha_0 \frac{\omega}{2^p}}\right](x,t)\, dx$$

$$\le \int_{B(R_0)} \ln^{+2}\left[\frac{\alpha_0 \frac{\omega}{2^s}}{\alpha_0 \frac{\omega}{2^s} - [\beta_n(u_n) - (\beta_n(\mu^+) - \alpha_0 \frac{\omega}{2^s})]^+ + \alpha_0 \frac{\omega}{2^p}}\right](x,\tau)\, dx$$

$$+\frac{C}{\sigma_1^2}\left(1+\ln 2^{p-s}\right)\left(1+\frac{R_0^{N\kappa}}{\alpha_0^2\left(\frac{\omega}{2^p}\right)^2}\right)\sup_{s\geq\beta_n^{-1}[\beta_n(\mu^+)-\alpha_0\frac{\omega}{2^s}]}\beta_n'(s)\,\kappa_N\,R_0^N\ .$$

We observe that $\beta_n(\mu^+) - \alpha_0\frac{\omega}{2^s} \geq \beta_n(\mu^+ - \frac{\omega}{2^s})$, and that $\beta_n'(s) \leq \beta'(s)$, $\forall s \in \mathbb{R}\backslash\{0\}$, $n = 1, 2, 3, \ldots$ Hence

$$\sup_{s\geq\beta_n^{-1}[\beta_n(\mu^+)-\alpha_0\frac{\omega}{2^s}]}\beta_n'(s) \leq \sup_{s\geq\mu^+-\frac{\omega}{2^s}}\beta'(s)\ .$$

If $\mu^+ - \frac{\omega}{2^s} \geq \delta_0$ (δ_0 is the number introduced in $[A_1]$) then $\sup_{s\geq\mu^+-\frac{\omega}{2^s}}\beta'(s) \leq \alpha_1$.

If $\mu^+ - \frac{\omega}{2^s} < \delta_0$, then by virtue of (4.2) and (4.3) $\mu^+ - \frac{\omega}{2^s} > \frac{\omega}{2^s}$. Hence, since $\frac{\omega}{2^s} < \delta \leq \delta_0$, in either case

$$\sup_{s\geq\mu^+-\frac{\omega}{2^s}}\beta'(s) \leq \beta'\left(\frac{\omega}{2^s}\right)\ .$$

Let $p_0 > p$ to be chosen, then if $\frac{\omega}{2^{p_0}} \geq R_0^{\frac{N\kappa}{2}}$, the last term in (4.6) can be majorized by

$$\beta'\left(\frac{\omega}{2^s}\right)\frac{2(1+\alpha_0^2)C}{\alpha_0^2\,\sigma_1^2}\ln 2^{p-s}\,\kappa_N\,R_0^N\ , \quad p \geq s+2\ .$$

We estimate the remaining terms in (4.6) as follows.

$$\int_{B(R)}\ln^{+2}\left[\frac{\alpha_0\frac{\omega}{2^s}}{\alpha_0\frac{\omega}{2^s}-[\beta_n(u_n)-(\beta_n(\mu^+)-\alpha_0\frac{\omega}{2^s})]^+ +\alpha_0\frac{\omega}{2^p}}\right](x,\tau)\,dx \leq \text{by Lemma 4.2} \leq$$

$$\leq (\ln 2^{p-s})^2\left(\frac{1-\theta_0}{1-\alpha}\right)\kappa_N\,R_0^N\ .$$

For the left-hand side of (4.6) we have

$$\int_{B(R-\sigma_1R)}\ln^{+2}\left[\frac{\alpha_0\frac{\omega}{2^s}}{\alpha_0\frac{\omega}{2^s}-[\beta_n(u_n)-(\beta_n(\mu^+)-\alpha_0\frac{\omega}{2^s})]^+ +\alpha_0\frac{\omega}{2^p}}\right](x,t)\,dx$$

$$\geq \int_{B(R-\sigma_1 R)\cap[\beta_n(u_n)>\beta_n(\mu^+)-\alpha_0\frac{\omega}{2^p}]} \ell n^{+2} \left[\frac{\alpha_0 \frac{\omega}{2^s}}{\alpha_0 \frac{\omega}{2^s} - [\beta_n(u_n) - (\beta_n(\mu^+) - \alpha_0 \frac{\omega}{2^s})]^+ + \alpha_0 \frac{\omega}{2^p}}\right](x,t)\,dx$$

$$\geq (\ell n^+ 2^{p-s-1})^2 \text{ meas } B^+_{\beta_n(\mu^+)-\alpha_0\frac{\omega}{2^p}, R-\sigma_1 R}(t)$$

where

$$B^+_{k,R}(t) \equiv \{x \in B(R) \mid \beta_n(u_n(x,t)) > k\} .$$

Combining the previous estimates in (4.6) we see that for all $t \in [\tau, t_0]$ we have

$$(4.7) \quad \text{meas } B^+_{\beta_n(\mu^+)-\alpha_0\frac{\omega}{2^p}, R_0-\sigma_1 R_0}(t)$$

$$\leq \left[\frac{\ell n\, 2^{p-s}}{\ell n\, 2^{p-s-1}}\right]^2 \left(\frac{1-\theta_0}{1-\alpha}\right) \kappa_N R_0^N + \frac{2C(1+\alpha_0^2)}{\alpha_0^2\sigma_1^2\, \ell n 2} \; \frac{p-s}{(p-s-1)^2}\; \beta'(\frac{\omega}{2^s})\; \kappa_N\, R_0^N .$$

Now

$$\text{meas } B^+_{\beta_n(\mu^+)-\alpha_0\frac{\omega}{2^p}, R_0}(t) \leq \text{meas } B^+_{\beta_n(\mu^+)-\alpha_0\frac{\omega}{2^p}, R_0-\sigma_1 R_0}(t)$$

$$+ \text{meas}\{B(R_0)\backslash B(R_0 - \sigma_1 R_0)\} \leq \text{meas } B^+_{\beta_n(\mu^+)-\alpha_0\frac{\omega}{2^p}, R_0-\sigma_1 R_0}(t) + N\, \sigma_1\, \kappa_N\, R_0^N .$$

Therefore from estimate (4.7) we obtain

$$\text{meas } B^+_{\beta_n(\mu^+)-\alpha_0\frac{\omega}{2^p}, R_0}(t) \leq \left\{\left(\frac{p-s}{p-s-1}\right)^2 \left(\frac{1-\theta_0}{1-\alpha}\right) + \frac{C_1}{\sigma_1^2}\; \beta'(\frac{\omega}{2^s})\; \frac{(p-s)}{(p-s-1)^2} + N\,\sigma_1\right\} \kappa_N\, R_0^N ,$$

where we have set in (4.7).

$$C_1 = \frac{2C(1+\alpha_0^2)}{\alpha_0^2\, \ell n\, 2}$$

This inequality holds for all $\sigma_1 \in (0,1)$, all $p \geq s+2$ and all $t \in [\tau, t_0]$.

Select

$$\sigma_1 = \frac{3}{8}\,\frac{\theta_0^2}{N}$$

and $p_0 \in \mathbb{N}$ so large that

$$\frac{C_1}{\sigma_1^2}\,\frac{p_0-s}{(p_0-s-1)^2}\,\beta'(\frac{\omega}{2^s}) < \frac{3}{8}\,\theta_0^2 \ , \quad \text{and}$$

$$\left(\frac{p_0-s}{p_0-s-1}\right)^2 \le (1-\alpha)(1+\theta_0) \ ,$$

to obtain

$$\text{meas } B^+_{\beta_n(\mu^+)-\alpha_0\frac{\omega}{2^p},R_0}(t) \le \left[1-(\frac{\theta_0}{2})^2\right]\kappa_N\,R_0^N = \left[1-\left(\frac{c_0\omega^b}{2}\right)^2\right]\kappa_N\,R_0^N$$

for all $t \in [t_0-\alpha\,R_0^2\,,\,t_0]$. The lemma is proved.

<u>Remarks</u>: (i) It is easily seen that a suitable choice of p_0 is

$$p_0 = s + 2 + \left[\frac{C_2}{(c_0\omega^b)^6}\,\beta'(\frac{\omega}{2^s})\right] \tag{4.8}$$

where

$$C_2 = 2^5 N^2 C_1$$

and $[a]$ denotes the largest integer contained in a .

(ii) The constants C_1 , C_2 depend only upon the data and not upon n , ω nor R_0 .

<u>Lemma 4.4</u>: There exists $p_* \in \mathbb{N}$, $p_* > p_0$ such that

$$\text{meas } A^+_{\mu^+-\frac{\omega}{2^{p_*}},R_0}(t) \le \left[1-(\frac{\theta_0}{2})^2\right]\kappa_N\,R_0^N$$

for all $t \in [\tau\,,\,t_0]$.

The number p_* depends upon ω but not upon n nor R_0 .

Proof of Lemma 4.4: Let $p_* = p_0 + r$, $r \in \mathbb{N}$ to be selected. We first establish the following fact. There exists $r \in \mathbb{N}$ (depending upon the data, and ω, but not upon n, nor R_0) such that

$$\beta_n(\mu^+) - \beta_n(\mu^+ - \frac{\omega}{2^{p_*}}) \leq \alpha_0 \frac{\omega}{2^{p_0}} .$$

Indeed by virtue of the smoothness of β_n

$$\beta_n(\mu^+) - \beta_n(\mu^+ - \frac{\omega}{2^{p_0+r}}) \leq \beta_n'(\xi) \frac{\omega}{2^{p_0+r}} ,$$

where $\mu^+ - \frac{\omega}{2^{p_0+r}} \leq \xi \leq \mu^+$. Now $\mu^+ - \frac{\omega}{2^{p_0+r}} > \mu^+ - \frac{\omega}{2^s} > \frac{\omega}{2^s}$ so that $\beta_n'(\xi) \leq \beta''\xi) \leq \beta'(\frac{\omega}{2^s})$.

To prove the claim it will suffice to select

$$r = 1 + \left[\log_2 \frac{\beta'(\frac{\omega}{2^s})}{\alpha_0} \right] .$$

Next we observe that since

$$\beta_n(\mu^+ - \frac{\omega}{2^{p_*}}) \geq \beta_n(\mu^+) - \alpha_0 \frac{\omega}{2^{p_0}}$$

$$\left[\beta_n(u_n) > \beta_n(\mu^+ - \frac{\omega}{2^{p_*}}) \right] \subseteq \left[\beta_n(u_n) > \beta_n(\mu^+) - \alpha_0 \frac{\omega}{2^{p_0}} \right] .$$

By monotonicity of β_n we have the inclusion

$$\left[u_n > \mu^+ - \frac{\omega}{2^{p_*}} \right] \subset \left[\beta_n(u_n) > \beta_n(\mu^+) - \alpha_0 \frac{\omega}{2^{p_0}} \right] ,$$

therefore lemma 4.4 follows from lemma 4.3.

Remarks: (i) Taking in account (4.8) and the previous argument we have the following expression for p_*.

$$p_* = s + 3 + \left[\frac{C_2}{(c_0\omega^b)^6} \beta'(\frac{\omega}{2^s}) \right] + \left[\log_2 \frac{\beta'(\frac{\omega}{2^s})}{\alpha_0} \right] \tag{4.9}$$

(ii) The result of Lemma 4.4 still holds if we replace p_* with any other number $q > p_*$.

Corollary 4.5: Let q be any positive integer larger than p_*. Then

$$\operatorname{meas}\left\{B(R_0)\setminus A^+_{\mu^+-\frac{\omega}{2^q},R_0}(t)\right\} \geq \left(\frac{\theta_0}{2}\right)^2 \kappa_N R_0^N$$

for all $t \in [t_0 - \alpha R_0^2, t_0]$.

Lemma 4.6: For any $\theta_1 > 0$ there exists $q_0 \in \mathbb{N}$, $q_0 > p_*$, such that if $\frac{\omega}{2^{q_0}} > R_0^{\frac{N\kappa}{2}}$, then

$$\operatorname{meas} Q^\alpha_{R_0}\left(\mu^+ - \frac{\omega}{2^{q_0}}\right) \leq \theta_1 \kappa_N R_0^{N+2}.$$

Proof of Lemma 4.6: Apply inequality (3.12) to the function $x \to u_n(x,t)$ in the ball $B(R_0) \times \{t\}$ for the levels

$$\ell = \mu^+ - \frac{\omega}{2^{q+1}}, \quad k = \mu^+ - \frac{\omega}{2^q}, \quad q_0 > q \geq p_*,$$

where q_0 has to be chosen. If we do this for all $t \in [t_0 - \alpha R_0^2, t_0]$ and take in account Corollary 4.5 we obtain the estimate

$$\left(\frac{\omega}{2^{q+1}}\right) \operatorname{meas}\left\{A^+_{\mu^+-\frac{\omega}{2^{q+1}},R_0}(t)\right\} \leq \frac{4D}{\theta_0^2\kappa_N} R_0 \int_{A^+_{k,R_0}(t)\setminus A^+_{\ell,R_0}(t)} |\nabla_x u_n| \, dx$$

$$\forall\, t \in [t_0 - \alpha R_0^2, t_0].$$

Integrate both the sides of this inequality over $[t_0 - \alpha R_0^2, t_0]$, square and use Hölder's inequality on the right-hand side, to obtain

$$(4.10)\quad \left(\frac{\omega}{2^{q+1}}\right)^2 \left[\operatorname{meas} Q_{R_0}\left(\mu^+ - \frac{\omega}{2^{q+1}}\right)\right]^2 \leq \left[\frac{4D}{\theta_0^2\kappa_N}\right]^2 R_0^2 \cdot$$
$$\left[\int_{t_0-\alpha R_0^2}^{t_0} \int_{A^+_{k,R_0}(\tau)\setminus A^+_{\ell,R_0}(\tau)} |\nabla_x u_n|^2 \, dx\,d\tau\right]\left[\int_{t_0-\alpha R_0^2}^{t_0} \operatorname{meas}[A^+_{k,R_0}(\tau)\setminus A^+_{\ell,R_0}(\tau)]\,d\tau\right]$$

$$\le \left[\frac{4D}{\theta_0^2\kappa_N}\right]^2 R_0^2 \left|(u_n - (\mu^+ - \frac{\omega}{2^q}))^+\right|^2_{V_2^{1,0}(Q_{R_0})} \left[\int_{t_0-\sigma R_0^2}^{t_0} \left[\text{meas } A^+_{k,R_0}(\tau)\setminus A^+_{\ell,R_0}(\tau)\right] d\tau\right] .$$

In order to estimate the $V_2^{1,0}(Q_{R_0})$-norm of $(u_n - (\mu^+ - \frac{\omega}{2^q}))^+$ we use inequalities (3.10) applied to the pair of cylinders Q_{R_0}, Q_{2R_0}. Notice that in this connection $\sup_{Q_{2R_0}} (u_n - (\mu^+ - \frac{\omega}{2^q}))^+ \le \frac{\omega}{2^q} < \delta$ and that

$$(\sigma_1 R_0)^{-2} = 4R_0^{-2} \; ; \; (\sigma_2 R_0^2)^{-1} = \frac{4}{3} R_0^{-2} .$$

Moreover observe that since $\mu^+ - \frac{\omega}{2^q} > 0$ the use of (3.10) is justified. Inequalities (3.10) now give

$$\left|(u_n - (\mu^+ - \frac{\omega}{2^q}))^+\right|^2_{V_2^{1,0}(Q_{R_0})} \le \gamma \sup_{s \ge \mu^+ - \frac{\omega}{2^q}} \beta'(s)[4 + \tfrac{4}{3}] R_0^{-2}$$

$$\cdot \|(u_n - (\mu^+ - \frac{\omega}{2^q}))^+\|^2_{2,Q_{2R_0}} + \gamma \left\{\int_{t_0-4R_0^2}^{t_0} \left[\text{meas } A^+_{\mu^+ - \frac{\omega}{2^q}, 2R_0}(\tau)\right]^{\frac{r}{q}} d\tau\right\}^{\frac{2}{r}(1+\kappa)}$$

$$\le \gamma \frac{2^{N+6}}{3} \left(\frac{\omega}{2^q}\right)^2 \sup_{s \ge \mu^+ - \frac{\omega}{2^q}} \beta'(s)\, \kappa_N R_0^N + \gamma\, 2^{N(1+\kappa)} \kappa_N^{\frac{2}{q}(1+\kappa)-1} R_0^{N\kappa} \kappa_N R_0^N$$

where (3.4) has been used. Since $\mu^+ - \frac{\omega}{2^q} > \mu^+ - \frac{\omega}{2^s} > \frac{\omega}{2^s}$ we have $\beta'(\mu^+ - \frac{\omega}{2^s}) \le \beta'(\frac{\omega}{2^s})$. Moreover by assumption

$$R_0^{N\kappa} \le \left(\frac{\omega}{2^{q_0}}\right)^2 < \left(\frac{\omega}{2^q}\right)^2 ,$$

so that there exists a constant C_3 depending only upon the dimension N and the data, such that

$$\left|(u_n - (\mu^+ - \frac{\omega}{2^q}))^+\right|^2_{V_2^{1,0}(Q_{R_0})} \le C_3\, \beta'\left(\frac{\omega}{2^s}\right)\left(\frac{\omega}{2^q}\right)^2 \kappa_N R_0^N .$$

Carrying this in (4.10) and dividing by $\left(\frac{\omega}{2^{q+1}}\right)^2$, gives

$$(4.11)\quad \left[\text{meas}\ Q^{\alpha}_{R_0}\left(\mu^+ - \frac{\omega}{2^{q+1}}\right)\right]^2 \le C_4\ \frac{\beta'\left(\frac{\omega}{2^s}\right)}{\theta_0^4}\ \kappa_N\ R_0^{N+2}$$

$$\cdot \left[\int_{t_0-\alpha R_0^2}^{t_0} \left[\text{meas}\ A^+_{k,R_0}(\tau)\setminus A^+_{\ell,R_0}(\tau)\right] d\tau\right]$$

where $C_4 = 4\ C_3\ [\frac{4D}{\kappa_N}]^2$.

We add inequalities (4.11) with respect to q , from p_* to $q_0 - 1$, and obtain

$$(q_0 - p_*)\left[\text{meas}\ Q^{\alpha}_{R_0}\left(\mu^+ - \frac{\omega}{2^{q_0}}\right)\right]^2 \le C_4\ \frac{\beta'\left(\frac{\omega}{2^s}\right)}{\theta_0^4}\ \kappa_N\ R_0^{N+2}$$

$$\sum_{q=p_*}^{q_0-1} \int_{t_0-\alpha R_0^2}^{t_0} \left[\text{meas}\ A^+_{\mu^+-\frac{\omega}{2^q},R_0}(\tau)\setminus A^+_{\mu^+-\frac{\omega}{2^{q+1}},R_0}(\tau)\right] d\tau$$

$$\le \frac{C_4}{2}\ \frac{\beta'\left(\frac{\omega}{2^s}\right)}{\theta_0^3}\ (\kappa_N\ R_0^{N+2})^2 .$$

Where we used the fact that $\alpha = \theta_0/2$.

Dividing the inequality by $q_0 - p_*$, to prove the lemma we have only to choose q_0 so large that

$$\frac{C_4\ \beta'\left(\frac{\omega}{2^s}\right)}{2(q_0-p_*)\ \theta_0^3} \le \theta_1^2 .$$

We will select

$$(4.12)\qquad q_0 = p_* + 1 + \frac{C_4\ \beta'\left(\frac{\omega}{2^s}\right)}{2\ \theta_0^3\ \theta_1^2} .$$

Consider now the pair of cylinders $Q^{\alpha}_{R_0}$ and

$$Q^{\alpha}_{\frac{R_0}{2}} \equiv \{|x-x_0| < \frac{R_0}{2}\} \times [t_0 - \alpha\frac{R_0^2}{4}\ ,\ t_0] .$$

For them we have the following result

Lemma 4.7: There exists a number $\theta_1 > 0$ depending upon α, N and the data, such that if

$$\text{meas } Q^{\alpha}_{R_0}\left(\mu^+ - \frac{\omega}{2^{q_0}}\right) < \theta_1 \kappa_N R_0^{N+2} ,$$

then either

(i) $$H = \operatorname*{ess\,sup}_{Q^{\alpha}_{R_0}} \left(u_n - \left(\mu^+ - \frac{\omega}{2^{q_0}}\right)\right)^+ \le R_0^{\frac{N\kappa}{2}} , \quad \text{or}$$

(ii) $$\text{meas } Q^{\alpha}_{\frac{R_0}{2}}\left(\mu^+ - \frac{\omega}{2^{q_0}} + \frac{1}{2} H\right) = 0 .$$

Proof of Lemma 4.7: The proof is very similar to the proof of lemma 3.1 of [10]. We reproduce the main steps mainly to trace the dependence of q_0 on θ_0 (and hence on ω).

Set

$$R_m = \frac{R_0}{2} + \frac{R_0}{2^{m+2}} , \quad \bar{R}_m = \frac{R_0}{2} + \frac{3R_0}{2^{m+4}}$$

and consider the cylinders

$$Q^{\alpha}_{R_m} \equiv \{|x - x_0| < R_m\} \times [t_0 - \alpha R_m^2, t_0]$$

$$\bar{Q}^{\alpha}_m \equiv \{|x - x_0| < \bar{R}_m\} \times [t_0 - \alpha R^2_{m+1}, t_0] ,$$

which satisfy the inclusions

$$Q^{\alpha}_{R_{m+1}} \subset \bar{Q}^{\alpha}_m \subset Q^{\alpha}_{R_m} .$$

We use inequalities (3.10) over $\bar{Q}^{\alpha}_m$ and $Q^{\alpha}_{R_m}$, for the functions $(x,t) \to (u_n - k_m)^+(x,t)$ where

$$k_m = \left(k_1 + \frac{1}{2} H\right) - \frac{1}{2^m} H , \quad m = 1,2,\ldots$$

$$k_1 = \mu^+ - \frac{\omega}{2^{q_0}} .$$

Since $k_m \geq k_1 > 0$ the use of inequalities (3.10) is justified.

Note that in this case $(\sigma_1 R_m)^{-2} = R_0^{-2}\, 2^{2(m+4)}$ and $(\sigma_2 \alpha R_m^2)^{-1} = R_0^{-2}\, \alpha^{-1}\, 2^{m+3}$. We have to show that the numbers

$$Y_m = \frac{1}{H^2 R_0^{N+2}}\, y_m = \frac{1}{H^2 R_0^{N+2}} \int_{Q^\alpha_{R_m}}\!\!\int (u_n - k_m)^{+2}\, dx d\tau$$

$$Z_m = \frac{z_m}{R_0^N} = \frac{1}{R_0^N}\left\{\int_{t_0-\alpha R_m}^{t_0} \left[\text{meas } A^+_{k_m,R_m}(\tau)\right]^{\frac{r}{q}} d\tau\right\}^{\frac{2}{r}}$$

tend to zero as $n \to \infty$. Proceeding exactly as in lemma 3.1 of [10], and using inequalities (3.10), we see that Y_m and Z_m satisfy the recursion inequalities

$$\text{[I]} \qquad Y_{m+1} \leq \frac{\tilde{C} 2^{4m}}{\alpha}\, \beta'\!\left(\frac{\omega}{2^s}\right) \left[Y_m^{1+\frac{2}{N+2}} + Y_m^{\frac{2}{N+2}} Z_m^{1+\kappa}\right]$$

$$\text{[II]} \qquad Z_{m+1} \leq \frac{\tilde{C} 2^{4m}}{\alpha}\, \beta'\!\left(\frac{\omega}{2^s}\right) \left[Y_m + Z_m^{1+\kappa}\right] .$$

where $\tilde{C}$ is a constant depending only upon the data and not upon n , ω nor R_0 . The procedure shows that $\tilde{C}$ can be taken to be so large that

$$\frac{\alpha}{2\tilde{C}} = \frac{c_0 \omega^b}{4\tilde{C}} < \frac{c_0 (2M)^b}{4\tilde{C}} < 1 .$$

By lemma 5.7 of [14] page 96, there exists a number $\lambda > 0$ such that if

$$Y_1 < \lambda \quad ; \quad Z_1 < \lambda^{\frac{1}{1+\kappa}} ,$$

then the recursion inequalities [I] - [II] imply that Y_m , $Z_m \to 0$ or $m \to \infty$. From [14], setting

$$d = \min\left\{\frac{2}{N+2}\, ;\, \frac{\kappa}{1+\kappa}\right\}$$

the number λ is given by

$$\lambda = \min\left\{\left[\frac{\alpha}{2\tilde{C}\beta'(\frac{\omega}{2^s})}\right]^{\frac{N+2}{2}} 2^{-\frac{5(N+2)}{2d}} \; ; \; \left[\frac{\alpha}{2\tilde{C}\beta'(\frac{\omega}{2^s})}\right]^{\frac{1+\kappa}{\kappa}} 2^{-\frac{5}{\kappa d}}\right\} .$$

Now since $\kappa = \frac{2\kappa_1}{N}$, $\kappa_1 \in (0,1)$ and $\frac{\alpha}{2\tilde{C}\beta'(\frac{\omega}{2^s})} < 1$ above gives

$$d = \frac{\kappa}{1+\kappa} \ ; \ \lambda \geq \lambda_0 = \left[\frac{\alpha}{2\tilde{C}\beta'(\frac{\omega}{2^s})}\right]^{\frac{1+\kappa}{\kappa}} \min\left[2^{-\frac{5(N+2)}{2d}} \ ; \ 2^{-\frac{5}{\kappa d}}\right]$$

$$= \sigma_0 \left[\frac{\theta_0}{\beta'(\frac{\omega}{2^s})}\right]^{\frac{1+\kappa}{\kappa}}$$

where we used the fact that $\alpha = \frac{\theta_0}{2}$ and we have set

$$\sigma_0 = (4\tilde{C})^{-\frac{1+\kappa}{\kappa}} \min\left[2^{-\frac{5(N+2)}{2d}} \ ; \ 2^{-\frac{5}{\kappa d}}\right].$$

The lemma follows if

$$Y_1 \leq R_0^{-(N+2)} \text{ meas } Q^{\alpha}_{R_0}(k_1) \leq \theta_1 \kappa_N$$

$$= \sigma_0 \left[\frac{\theta_0}{\beta'(\frac{\omega}{2^s})}\right]^{\frac{1+\kappa}{\kappa}} = \sigma_0 \left[\frac{\theta_0}{\beta'(\frac{\omega}{2^s})}\right]^{b}$$

From (4.12) and the remarks above it follows that the conclusion of lemma 4.7 holds true if we choose

$$q_0 = p_* + 1 + \left[\frac{\kappa_N^2 C_4 \ \beta'(\frac{\omega}{2^s})^{1+2b}}{2\ \sigma_0^2\ \theta_0^{3+2b}}\right]. \tag{4.13}$$

We now recall the definition of p_* and that $\theta_0 = c_0\omega^b$, and deduce that a suitable choice of q_0 is given by

$$q_0 = s + 5 + \left[\left(\frac{\kappa_N^2 C_4}{2\ \sigma_0^2} + C_2\right) c_0^{-\max\{6;2b+3\}} \cdot \frac{(\beta'(\frac{\omega}{2^s}))^{1+2b}}{\omega^{b\max\{6;2b+3\}}}\right] + \left[\log_2 \frac{\beta'(\frac{\omega}{2^s})}{\alpha_0}\right].$$

Set

$$A = \left(\frac{\kappa_N^2 C_4}{2\,\alpha_0^2} + C_2\right) c_0^{-\max\{6;2b+3\}}$$

$$a = b\max\{6;2b+3\}$$

$$c = 1 + 2b \ .$$

Then

$$q_0 = s + 5 + \left[A\,\frac{\beta'(\frac{\omega}{2^s})^c}{\omega^a}\right] + \left[\log_2 \frac{\beta'(\frac{\omega}{2^s})}{\alpha_0}\right] .$$

We remark that s, A, a, c depend only upon the data and not upon n, ω nor R_0.

<u>Proof of the Proposition</u>: Set

$$\tau(\omega) = s + 5 + A\,\frac{\left[\beta'(\frac{\omega}{2^s}) + \beta'(-\frac{\omega}{2^s})\right]^c}{\omega^a} + \log_2\left[\frac{\beta'(\frac{\omega}{2^s}) + \beta'(-\frac{\omega}{2^s})}{\alpha_0}\right] ,$$

and suppose that

$$\frac{\omega}{2^s} > \frac{\omega}{2^{q_0+1}} \ge \frac{\omega}{2^{\pi(\omega)}} > (2R_0)^{\frac{N\kappa}{2}} . \tag{4.14}$$

Obviously either

1. $\operatorname{meas} Q^-_{R_0}(\mu^- + \frac{\omega}{2^s}) \le c_0\,\omega^{\frac{2\kappa_1}{N+2\kappa_1}}\,\kappa_N\,R_0^{N+2}$.

or

2. $\operatorname{meas} Q^-_{R_0}(\mu^- + \frac{\omega}{2^s}) > c_0\,\omega^{\frac{2\kappa_1}{N+2\kappa_1}}\,\kappa_N\,R_0^{N+2}$

<u>Case 1</u>: By lemma 4.1 either

1.a. $\sup\limits_{Q_{R_0}} (u_n - (\mu^- + \frac{\omega}{2^s}))^- \le R_0^{\frac{N\kappa}{2}}$, or

1.b. meas $Q^{-}_{\frac{R_0}{2}}(\mu^- + \frac{\omega}{2^s} - \frac{1}{2}H) = 0$.

If 1.a occurs then

$$-\inf_{Q_{R_0}} u_n \le -\mu^- - \frac{\omega}{2^s} + R_0^{\frac{N\kappa}{2}} \le \text{by (4.14)} < -\mu^- - \frac{\omega}{2^{\pi(\omega)}} .$$

Adding $\sup_{Q_{R_0}} u_n$ on the left-hand side and μ^+ on the right-hand side we obtain

$$\operatorname{osc}_{Q_{R_0}} u_n < \operatorname{osc}_{Q_{2R_0}} u_n - \frac{\omega}{2^{\pi(\omega)}} \le \omega\left(1 - \frac{1}{2^{\pi(\omega)}}\right) .$$

If 1.b occurs then

$$-\inf_{Q_{\frac{R_0}{2}}} u_n \le -\mu^- - \frac{\omega}{2^s} + \frac{1}{2}(-\mu^- + \frac{\omega}{2^s} + \mu^-)$$

$$= -\mu^- - \frac{\omega}{2^{s+1}}$$

i.e.

$$\operatorname{osc}_{Q_{\frac{R_0}{2}}} u_n \le \omega\left(1 - \frac{1}{2^{\pi(\omega)}}\right).$$

Case 2: By Lemmas 4.2 - 4.6 in view of (4.14), the assumptions of lemma 4.7 are verified, It gives the following alternative. Either

$$\sup_{Q^{\alpha}_{\frac{R_0}{2}}} u_n \le \mu^+ - \frac{\omega}{2^{q_0}} + R_0^{\frac{N\kappa}{2}} \le \mu^+ - \frac{\omega}{2^{q_0+1}}$$

or

$$\operatorname{ess\,sup}_{Q^{\alpha}_{\frac{R_0}{2}}} u_n \le \mu^+ - \frac{\omega}{2^{q_0}} + \frac{\omega}{2^{q_0+1}} = \mu^+ - \frac{\omega}{2^{q_0+1}} .$$

Hence in either case

$$\operatorname*{osc}_{Q^{\alpha}_{\frac{R_0}{2}}} u_n \le \omega\left(1 - \frac{1}{2^{\pi(\omega)}}\right) .$$

Now to determine R_* notice that by virtue of (4.14)

$$\alpha\left(\frac{R_0}{2}\right)^2 = \frac{1}{2 \cdot 4^2} c_0\omega^b (2R_0)^2 > 2^{sb-5} c_0 (2R_0)^{2+\frac{N\kappa b}{2}}$$

Setting $\xi_*^2 = \min\{1 ; 2^{sb-5} c_0\}$, and

$$h = 1 + \frac{N\kappa b}{4}$$

we have

$$\sqrt{\alpha}\,\frac{R_0}{2} \ge \xi_*(2R_0)^h \equiv R_* ,$$

so that $Q^{\alpha}_{\frac{R_0}{2}} \supset Q_{R_*} \equiv \{|x - x_0| < R_*\} \times [t_0 - R_*^2 , t_0]$.

It follows that

$$\operatorname*{osc}_{Q_{R_*}} u_n \le \omega\left(1 - \frac{1}{2^{\pi(\omega)}}\right) \tag{4.15}$$

Finally if (4.3) is false then (4.15) follows at once. The proposition is proved.

5. Proof of Theorem 1:

Proposition 4.1 holds true for any number ω satisfying

$$\operatorname*{osc}_{Q_{2R_0}} u_n \le \omega \le 2M$$

We stress the fact that the constants ξ_* , a , b , c , A , s , h in proposition 4.1 do not depend upon n , ω nor R_0 . Let $(x_0 , t_0) \in \Omega_T$ be fixed and select

$$\omega = 2M \geq \operatorname*{osc}_{Q_{\Omega_T}} u_n .$$

Let $0 < R_0 < \frac{1}{2}$ be so small that $Q_{2R_0} \subset \Omega_T$ and

$$(2R_0)^{\frac{N\kappa}{2}} \leq \frac{2M}{2^{\pi(2M)}} . \tag{5.1}$$

Define sequences of positive real numbers $\{M_m\}$, $\{\xi_m\}$, $\{R_m\}$ as follows

$$M_1 = 2M \; ; \; M_{m+1} = M_m \left(1 - 2^{-\pi(M_m)}\right) , \; m = 2 , 3, \ldots$$

$$\xi_m = \min \left\{ \xi_* \; ; \; \left[\frac{2^{\pi(M_m)}}{2^{\pi(M_{m+1})}} \right]^{\frac{2}{N\kappa}} \right\}$$

$$R_1 = 2R_0 \; ; \; R_{m+1} = \xi_m (R_m)^h \; ; \; m = 2 , 3, \ldots$$

<u>Lemma 5.1</u>: $\{M_m\} \searrow 0$; $\{R_m\} \searrow 0$ and for all $m \in \mathbb{N}$

$$\operatorname*{osc}_{Q_{R_m}} u_n \leq M_m$$

<u>Proof of Lemma 5.1</u>: If $M_m > M_{m+1} > \ldots > M_0 > 0$, then for all $m \in \mathbb{N}$ $M_{m+1} \leq M_m \varepsilon$; $\varepsilon = [1 - 2^{-\pi(M_0)}] < 1$. Therefore $M_m \leq M_0 \varepsilon^m$, $m = 1 , 2, \ldots$, which implies $M_m \searrow 0$ as $m \to \infty$. A contradiction. The second statement is obvious.

In view of (5.1), proposition 4.1 implies that

$$\operatorname*{osc}_{Q_{R_2}} u_n \leq M_2 .$$

Moreover

$$(R_2)^{\frac{N\kappa}{2}} = \xi_1^{\frac{N\kappa}{2}} \left(R_1^{\frac{N\kappa}{2}}\right)^h \leq \xi_1^{\frac{N\kappa}{2}} \left[M_1 2^{-\pi(M_1)}\right]^h .$$

Using the definition of M_2 we have

$$(R_2)^{\frac{N\kappa}{2}} \le \xi_1^{\frac{N\kappa}{2}} \left[\frac{M_2}{2^{\pi(M_1)}\left(1 - 2^{-\pi(M_1)}\right)} \right]^{b} < \xi_1^{\frac{N\kappa}{2}} \frac{M_2}{2^{\pi(M_2)}} \frac{2^{\pi(M_2)}}{2^{\pi(M_1)}} \, 2 \left(\frac{2M_2}{2^{\pi(M_1)}} \right)^{\frac{N\kappa b}{4}}$$

Recalling the definition of b and s, it is immediate to see that

$$2 \left(\frac{2M_2}{2^{\pi(M_2)}} \right)^{\frac{N\kappa b}{2}} < 1 \, .$$

Moreover $\xi_1^{\frac{N\kappa}{2}} \le \frac{2^{\pi(M_1)}}{2^{\pi(M_2)}}$; therefore,

$$(R_2)^{\frac{N\kappa}{2}} \le \frac{M_2}{2^{\pi(M_2)}} \, .$$

Thus we have shown that the two inequalities

$$\operatorname*{osc}_{Q_{R_1}} u_n \le M_1 \qquad\qquad R_1^{\frac{N\kappa}{2}} \le M_1 \, 2^{-\pi(M_m)} \, ,$$

imply the same inequalities for R_2 and M_2 . The same argument shows that if

$$\operatorname*{osc}_{Q_{R_m}} u_n \le M_m \qquad\qquad R_m^{\frac{N\kappa}{2}} \le M_m \, 2^{-\pi(M_m)}$$

then the same inequalities are valid for $m + 1$. The lemma is proved.

We remark that the construction of the sequences $\{M_m\}$ and $\{R_m\}$ depends only upon the data and is independent of the properties of the approximations $u_n(x,t)$.

<u>Lemma 5.2</u>: *If* K *is a compact contained in* Ω_T *, then there exists a non-decreasing continuous function* $\omega_K(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ *,* $\omega_K(0) = 0$ *, such that*

$$|u_n(x_1, t_1) - u_n(x_2, t_2)| \le \omega_K(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}})$$

$\forall (x_i, t_i) \in K$, $i = 1, 2$ and $\forall n \in \mathbb{N}$.

The function $\omega_K(\cdot)$ is determined only in terms of the data and dist $(K;\Gamma)$.

the statement is a consequence of Lemma 5.1 and establishes theorem 1.

6. Continuity at Γ

In this section we state the assumptions that permit at least in some case, to extend the interior continuity of the solution of (1.1) up to the parabolic boundary of Ω_T.

[A] - Continuity at $t = 0$

Let $u \in V_2(\Omega_T)$ be a weak solution of (1.1) satisfying the identity

$$(6.1) \quad \int_\Omega \beta(u(x,t))\,\phi(x,t)dx + \int_0^t \int_\Omega \{-\beta(u)\,\phi_t + \vec{a}(x,\tau,u,\nabla_x u)\cdot\nabla_x\phi$$
$$+ b(x,\tau,u,\nabla_x u)\,\phi\}\,dxd\tau = \int_\Omega \beta(u_0(x))\,\phi(x,0)\,dx$$

for all $\phi \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ and a.e. $0 \le t \le T$, where $u_0(x) \in L_2(\Omega)$ and is continuous over a compact $K \subset \Omega$ with modulus of continuity $\omega_{0,K}(\cdot)$. On $u(x,t)$ we assume the following

$[A_1]$ $u(x,t)$ satisfies $[A_4]$ where the functions $u_n(x,t)$ satisfy also identify (6.1) with $\beta(\cdot)$ replaced by $\beta_n(\cdot)$, and all $0 \le t \le T$.

Theorem 6.1: The functions $(x,t) \to u_n(x,t)$ are equicontinuous on $K' \times [0,T]$, for every compact $K' \subset K$

Corollary: There exists a non-decreasing continuous function $s \to \omega(s) : \mathbb{R}^+ \to \mathbb{R}^+$, $\omega(0) = 0$ such that

$$|u(x_1, t_1) - u(x_2, t_2)| \le \omega(|x_1 - x_2| + |t_1 - t_2|^{1/2})$$

for all $(x_i, t_i) \in K' \times [0,T]$, $i = 1, 2$, and all compacts $K' \subset K$.

The function $s \to \omega(s)$ depends uniquely upon the data and the modulus of continuity of $u_0(x)$ over K.

[B] The case of Neumann boundary data

Consider (formally) the problem

$$(6.2)\quad \begin{cases} \frac{\partial}{\partial t}\beta(u) - \operatorname{div} \vec{a}(x,t,u,\nabla_x u) + b(x,t,u,\nabla_x u) = 0 \\ \vec{a}(x,t,u,\nabla_x u)\cdot \vec{n}_{S_T}(x,t) = g(x,t,u) \quad \text{on} \quad S_T \\ u(x,0) = u_0(x) \quad x\in\Omega\ , \quad u_0(x)\in L_2(\Omega)\ . \end{cases}$$

where $\vec{n}_{S_T}(x,t) = (n_{x_1}, n_{x_2}, \ldots, n_{x_N})$ denotes the outer unit normal to S_T .

On the boundary data $g(x,t,u)$ *we assume that*

[G] g is continuous over $S_T \times \mathbb{R}$ and admits an extension $\tilde{g}(x,t\ ,u(x,t))$ over Ω_T such that

$$\left\|\frac{\partial}{\partial u}\tilde{g}(x,t,u(x,t))\ ,\ \frac{\partial}{\partial x}\tilde{g}(x,t,u(x,t))\right\|_{\infty,\Omega_T} \le C < \infty\ .$$

By a weak solution of (6.2) we mean a function $u\in V_2(\Omega_T)$ satisfying

$$(6.3)\quad \int_\Omega \beta(u(x,t))\,\phi(x,t)\,dx + \int_0^t\int_\Omega \{-\beta(u)\,\phi_t + \vec{a}(x,\tau,u,\nabla_x u)\cdot\nabla_x\phi$$
$$+ b(x,\tau,u,\nabla_x u)\phi\}\,dxd\tau = \int_0^t\int_{\partial\Omega} g(x,\tau,u(x,\tau))\,\phi(x,\tau)\,d\sigma$$
$$+\int_\Omega \beta(u_0(x))\,\phi(x,0)\,dx$$

for all

(A_2) We assume that $u(x,t)$ can be constructed as the weak $V_2(\Omega_T)$ limit of a sequence of $u_n(x,t)\in V_2^{1,0}(\Omega_T)$ where u_n satisfy identity (6.3) with $\beta(\cdot)$ replaced by $\beta_n(\cdot)$ and $u(x,t)$ replaced by $u_n(x,t)$. Moreover we assume that the sequence $\{u_n\}$ is equibounded, i.e.

$$\|u_n\|_{\infty,\Omega_T} \le M \quad , \quad n = 1\ ,2\ ,3,\ldots$$

for a fixed positive number M .

<u>Theorem 6.2</u>: Assume that $\partial\Omega$ is a C^1 manifold in $\mathbb{R}^{N-1}$, and suppose that [G] holds. Then the sequence $\{u_n\}$ is equicontinuous in $\bar{\Omega}\times[\tau,T]$, for every $\tau > 0$.

If in addition $u_0(x)$ is continuous in $\bar{\Omega}$, then the sequence $\{u_n\}$ is equicontinuous in $\bar{\Omega}_T$.

Corollary: Let $u \in V_2(\Omega_T)$ be a weak-solution of (6.3), let (A_2) and [G] hold, and assume that $\partial\Omega$ is a C^1 manifold in $\mathbb{R}^{N-1}$. Then for every $\tau > 0$ there exists $\omega_\tau(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, $\omega_\tau(0) = 0$ continuous and non-decreasing such that

$$|u(x_1, t_1) - u(x_2, t_2)| \le \omega_\tau(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}})$$

$$\forall (x_i, t_i) \in \bar{\Omega} \times [\tau, T] .$$

If in addition $u_0(x)$ is continuous in $\bar{\Omega}$, then $(x,t) \to u(x,t)$ is continuous over $\bar{\Omega}_T$ with modulus of continuity $\omega_0(\cdot)$. The function $\omega_\tau(\cdot)$ can be determined in terms of the data and the number $\tau > 0$, whereas $\omega_0(\cdot)$ depends on the data and the modulus of continuity of $u_0(x)$ on $\bar{\Omega}$.

[C] The case of homogeneous Dirichlet boundary data:

Let $u \in V_2(\Omega_T)$ be a weak solution of (1.1) which in addition satisfies

(6.4) $\quad u|_{S_T} = 0 , (x,t) \in S_T$ a.e. $t \in [0,T]$

in the sense of the traces over S_T.

On $\partial\Omega$ assume the following

(P) $\exists\ \theta^* > 0$, $R_0 > 0$ such that $\forall x_0 \in \partial\Omega$ and every ball $B(R)$ centered at x_0, $R \le R_0$

$$\text{meas}[\Omega \cap B(R)] < (1-\theta^*) \text{ meas } B(R) .$$

Moreover on u impose the assumption

(A_3) $u(x,t)$ satisfies $[A_4]$ where each of the u_n has zero trace on S_T

Theorem 6.3: Let (6.4), (P) and (A_3) hold. Then the sequence $\{u_n\}$ is equicontinuous in $\bar{\Omega} \times [\tau, T]$, for all $\tau > 0$.

If in addition $u(x,t)$ satisfies (6.1) for all $\phi \in W_2^{1,1}(\Omega_T)$ with $u_0(x)$ continuous over $\bar{\Omega}$, then the sequence $\{u_n\}$ is equicontinuous in $\bar{\Omega}_T$.

Corollary: Let $u \in V_2(\Omega_T)$ be a weak solution of (1.1) satisfying (6.4) and let (P) and (A_3) hold. Then for every $\tau > 0$ there exists a continuous non-decreasing function $\omega_\tau(\cdot)$, $\omega_\tau(0) = 0$ such that

$$|u(x_1, t_1) - u(x_2, t_2)| \le \omega_\tau(|x_1 - x_2| + |t_1 - t_2|^{1/2})$$

for all $(x_i, t_i) \in \bar{\Omega} \times [\tau, T]$, $i = 1, 2$.

If in addition $u(x,t)$ satisfies (6.1) for all $\phi \in W_2^{1,1}(\Omega_T)$ with $u_0(x)$ continuous over $\bar{\Omega}$, then $(x,t) \to u(x,t)$ in continuous over $\bar{\Omega}_T$ with modulus of continuity $\omega_0(\cdot)$.

The functions $\omega_\tau(\cdot)$ can be determined in terms of the data and the numbers τ and θ^* of (P), whereas $\omega_0(\cdot)$ can be determined in terms of the data, θ^* and the modulus of continuity of $u_0(x)$ over Ω.

Remark: In [10] for the case of homogeneous Dirichlet boundary data, we derived a modulus of continuity of Hölder type, for the solution $u(x,t)$, near the lateral boundary S_T. This is not the case in the present situation because of the different nature of the graph $\beta(\cdot)$.

The proof of Theorems 6.1 - 6.3 can be given by using the same method of proof for the analogous theorems 5.1 - 5.3 of [10], modulo some modifications due to the nature of β.

It should be pointed out that we were unable to give an answer for the case of non-homogeneous and continuous Dirichlet boundary data.

7. Closing remarks

I. The fast diffusion case

Consider formally the equation

$$\frac{\partial}{\partial t} u - \operatorname{div} \vec{a}(x,t,u,\nabla_x \beta(u)) + b(x,t,u,\nabla_x u) = 0 \tag{7.1}$$

in Ω_T, where $s \to \beta(s)$ is a monotone function satisfying assumption $[A_1]$. We assume that the vector $\vec{a}(x,t,u,\vec{p})$ and the function $b(x,t,u,\vec{p})$ satisfy the same growth conditions stated in $[A_2]$ - $[A_3]$.

By a weak solution of (7.1) in Ω_T we mean a function $u \in V_2(\Omega_T)$, such that $|\nabla_x \beta(u)| \in L_2(\Omega_T)$, and satisfying

$$(7.2) \qquad \int_\Omega u\varphi dx\Big|_{t_0}^{t} + \int_{t_0}^{t}\int_\Omega \{-u\varphi_t + \vec{a}(x,\tau,u,\nabla_x\beta(u)) \cdot \nabla_x\varphi + b(x,\tau,u,\nabla_x u)\varphi\}dxd\tau = 0$$

for all $\varphi \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$ and almost all $0 < t_0 < t < T$.

Our motivation for considering such an equation is that (7.1) represents a quasi-linear generalization of

$$(7.3) \qquad u_t = \Delta(|u|^m \text{sign } u) \quad \text{in} \quad \mathcal{D}'(\Omega_T)$$

for the case $0 < m < 1$.

Equation (7.3) arises as a model for the spatial spread of biological population or in some problems in plasma physics [17, 18].

Our purpose here is to point out that weak solutions of (7.1) are continuous in their domain of definition, regardless of their signum. That is the continuity is a consequence solely of the local smoothing effect of operators like in (7.1).

Let $\beta_n(\cdot)$, denote a sequence of $C^\infty(\mathbb{R})$ functions such that $\beta_n \to \beta$ uniformly on compacts of $\mathbb{R}\backslash\{0\}$ and satisfying (2.2).

$[A_4]$ Let $u \in V_2(\Omega_T)$ be an essentially bounded weak solution of (7.1). We assume that u can be constructed as the weak $V_2(\Omega_T)$ - limit of a sequence $\{u_n\}$ such that

$$\|u_n\|_{\infty,\Omega_T} \le M < \infty \quad , \quad n = 1,2,3,\ldots$$

for a positive constant M, and each $u_n \in V_2^{1,0}(\Omega_T)$ is a weak solution of (7.1) in the sense of identity (7.2), with $\beta(\cdot)$ replaced by $\beta_n(\cdot)$.

<u>Theorem 7.1</u>: Let $u \in V_2(\Omega_T)$ be a weak solution of (7.1) satisfying $[A_4]$. Then $(x,t) \to u(x,t)$ is continous in Ω_T.

We leave to the reader the task of stating facts about boundary regularity along the lines of section 6.

The proof of Theorem 7.1 can be given by straightforward modifications of the methods of [10] and the ones in section 4 - 5 here.

We omit the details.

Corollary: Let $u \in V_2(\Omega_T)$ be a weak solution of

$$u_t = \Delta(|u|^m \text{sign } u) \quad \text{in} \quad \mathcal{D}'(\Omega_T)$$

$$m > 0 \quad .$$

Then $(x,t) \to u(x,t) \in C(\Omega_T)$.

II. On the Harnack inequality

Let $u \in V_2(\Omega_T)$ be an essentially bounded non-negative weak solution of (1.1) with $\beta(\cdot)$ replaced by the identity graph. Then it satisfies the Harnack inequality in the form introduced by Moser [19]. To be specific let $(x_0,t_0) \in \Omega_T$ and let R be so small that Q_R of "vertex" (x_0,t_0) is contained in Ω_T. For any given set of numbers $\sigma_i \in (0,1)$, $i = 1,2,3,4$ one can find a constant $\gamma = \gamma(\sigma_i)$ such that

$$\inf_{R^+} u \geq \gamma \sup_{R^-} u \tag{7.4}$$

where R^+, R^- are the cylinders

$$R^+ \equiv B(R - \sigma_1 R) \times [t_0 - \sigma_2 R^2, t_0] \quad , \quad \sigma_2 > \sigma_4 > \sigma_3 \quad ,$$

$$R^- \equiv B(R - \sigma_1 R) \times [t_0 - \sigma_3 R^2, T_0 - \sigma_4 R^2] \quad .$$

Inequality (7.4) was proved by Moser [19] for the linear case and generalized by several authors [20, 21] to the full quasi-linear situation. It turns out that (7.4) can be used to prove the Hölder continuity of essentially bounded weak solution of (1.1) with $\beta(s) = s$.

Here we want to give examples which show that essentially bounded non-negative weak solutions of equations like (1.1) or (7.1) do not, in general, satisfy the Harnack inequality.

(a) case $m > 1$:

Consider the Cauchy problem

$$(7.5)\qquad \begin{cases} (u^{\frac{1}{m}})_t - \Delta u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R} \times \mathbb{R}^+) \\ u(x,0) = (\cos x)^{2\frac{m}{m-1}} \quad x \in \mathbb{R} . \end{cases}$$

Aronson [3] shows that (7.5) admits a global unique weak solution, which is classical for $0 \leq t < T = (m-1)/2m(m+1)$. Moreover setting $x_j = (2j - 1)\frac{\pi}{2}$, $j \in \mathbb{Z}$, we have

$$u(x_j,t) = 0 \quad , \quad 0 \leq t < T$$

$$u(x,t) > 0 \quad , \quad \text{on } \mathbb{R} \setminus \bigcup_{j\in \mathbb{Z}} \{x_j\} \times [0,T) .$$

Fix the point $(x_j,T-\varepsilon)$, $\varepsilon > 0$ and let R be so small that the cylinder Q_R of "vertex" $(x_j\ t-\varepsilon)$, is contained in $\mathbb{R} \times (0,T)$. Then for such a choice, inequality (7.4) fails to be satisfied.

(b) case $0 < m \leq 1$:

Consider the Cauchy problem

$$(7.6)\qquad \begin{cases} u_t = \Delta(|u|^m \text{sign } u) \quad \text{in } \mathcal{D}'(\mathbb{R}^N \times (0,T)) \\ u(\cdot,0) = u_0(\cdot) \\ u_0 \geq 0 \quad , \quad u_0 \not\equiv 0 \\ u_0 \in L_1(\mathbb{R}^N) \cap L_\infty(\mathbb{R}^N) . \end{cases}$$

If $0 < m \leq (N-2)^+/N$, then (7.6) admits a unique, bounded non negative solution, which vanishes identically after a finite time $T = T(m)$. Hence the Harnack inequality is not satisfied.

If $(N-2)^+/N < m \leq 1$, then (7.6) admits a unique, bounded solution which is strictly positive over $\mathbb{R}^N \times (0,\infty)$. Consequently by the results of [20] - [21] it satisfies (7.4), where γ depends upon a local lower and upper bound on u.

For the stated existence-uniqueness results we refer to [15] and the extensive bibliography on it.

III. Extensions

The methods used in sections 3 - 5 might be modified to include graphs β having a singularity of different nature with respect to the one considered here, provided the coercivity is kept.

Discussing the results with Professor W. Ziemer, he pointed out that for example graphs like

$$\beta(s) = \ln|s|$$

could be handled as well.

Acknowledgement

I would like to thank Michel Pierre, Mike Crandall and William Ziemer, for helpful discussions and advice.

REFERENCES

[1] D.G. Aronson, Regularity properties of flows through porous media, SIAM J. Appl. Math., 17 (1969) 461-467.

[2] D.G. Aronson, Regularity properties of flows through porous media. The interface, Arch. Rat. Mech. Anal., 37 (1970) 1-10.

[3] D.G. Aronson, Regularity properties of flows through porous media. A counterexample. SIAM J. Appl. Math. 19 (1970) 299-307.

[4] D.G. Aronson, Ph. Benilan, Régularité des solutions de l'équation des milieux poreux dans $\mathbb{R}^N$, C.R. Acad. Sc. Paris, 288 (1979) 103-105.

[5] L.A. Caffarelli and A. Friedman, Continuity of the density of a gas flow in a porous medium, Trans. Amer. Math. Soc. 252 (1979) 99-113.

[6] L.A. Caffarelli and A. Friedman, Regularity of the free-boundary of a gas flow in an n-dimensional porous medium, Indiana Univ. J. of Math. Vol. 29 #3 (1980).

[7] L.A. Caffarelli and C.L. Evans, Continuity of the temperature in the two-phase Stefan problem (to appear).

[8] E. DeGiorgi, Sulla differenziabilità e l'analiticitá delle estremali degli integrali multipli regolari, Mem. Accad. Sci. Torino. Cl. Sc. Fis. Mat. Nat. (3) 3 (1957).

[9] E. DiBenedetto, Regularity results for the porous media equation. Annali Mat. Pura ed Appl. (IV), Vol. CXXI, pp. 249-262.

[10] E. DiBenedetto, Continuity of weak solutions to certain singular parabolic equations. Mathematics Research Center, technical Summary report #2124 (1980).

[11] B.H. Gilding, Hölder continuity of solutions of parabolic equations, J. London Math. Soc. (2), 13 (1976) 103-106.

[12] A.S. Kalashnikov, On the differential properties of generalized solutions of equations of the non-steady-state filtration type, Vestnik Moskovskogo Universiteta Matematica, 29 #1 (1974) pp. 62-68 (Russian).

[13] S.N. Kružkov, Results concerning the nature of the continuity of solutions of parabolic equations and some of their applications. Matematicheskie Zametki, Vol. 6, #1, (1969).

[14] O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'tzeva, Linear and quasi-linear equations of parabolic type. AMS Transl. Mono. 23, Providence, R.I. (1968).

[15] L.A. Peletier, The porous media equation. Notes University of Leiden, The Netherlands.

[16] P. Sacks, Existence and regularity of solutions of the inhomogeneous porous medium equation. Mathematics Research Center,

[17] M. E. Gurtin and R. C. MacCamy, On the diffusion of biological populations, Mathematical Biosciences, 33 (1977) 35-49.

[18] J. C. Berryman, Evolution of a stable profile for a class of nonlinear diffusion equations with fixed boundaries, J. Math. Phys. 18 (1977) 2108-2115.

[19] J. Moser, A Harnack inequality for Parabolic Differential Equations, Comm. Pure and Applied Math. Vol XVII, 101-134 (1964).

[20] D. G. Aronson and J. Serrin, Local behaviour of solutions of quasilinear parabolic equations, Arch. Rat. Mech. Anal. Vol 25, 1967, 81-123.

[21] N. S. Trundinger, Pointwise estimates and Quasilinear Parabolic Equations, Comm. Pure and Applied Math. Vol XXI, 205-226 (1968).

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2189	2. GOVT ACCESSION NO. AD-A099351	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Continuity of Weak Solutions to a General Porous Media Equation		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Emmanuele DiBenedetto		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE March 1981
		13. NUMBER OF PAGES 46
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

singular or degenerate evolution equation, free boundary, porous media

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We establish the continuity of weak solutions to singular equations of the type

$$\frac{\partial}{\partial t} \beta(u) - \operatorname{div} \vec{a}(x,t,u,\nabla_x u) + b(x,t,u,\nabla_x u) = 0 ,$$

where $\beta(\cdot)$ is a graph satisfying assumptions appropriate for the equation of porous media, in particular for the filtration of gases.

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